Journal of Computing and Information Technology - CIT 3, 1995, 4, 255–277

Discriminators in Lambda Calculus

Svitan Gaborovič

Faculty of Civil Engineering, University of Maribor, Maribor, Slovenia

The paper treats the problems of discriminability and separability for a wide class of infinite sets of terms in the pure lambda calculus. The technique used is a generalization of the Böhm out technique for extracting substitutional instances from the sets of terms. A unification of this technique on an appropriate class of terms enables the construction of an algorithm of extraction which leads to the construction of a discriminator for an infinite set of terms having the distinctive paths. In [11] some conditions under which a r.e. set of terms is a numeral system were done. For the opposite direction, i.e. construction of the distinctive paths for the discriminable or separable set of terms, we used the sequentiality and continuity theorem with a combination of a slight generalization of the Wadsworth's version of the $\lambda\bot$ calculus (cf. [8]). The connection between the semantic notion of discriminability and the syntactic notion of distinctiveness, as known in the case of finite sets of terms, is extended here to the class of terms on which the (generalized) Böhm out technique can be applied.

Keywords: Lambda calculus, lambda term, discriminator, separability of terms, discriminability, numeral

1. Introduction

The problem of separability of lambda terms was completely solved 1978 by Coppo et all. in [8] for finite sets of λ–terms. The case of infinite sets was studied by Ronchi della Rocca in [11], and some interesting cases of numeral systems were treated by Wadsworth in [12]. Our main interest is in generalizing the Böhm-out technique to treat the connection between the semantic notion of separability and syntactic notion of distinctiveness – analogously as in the finite case was done by Coppo et all. Together with the generalization of the Böhm-out technique, the *sequentiality* and *continuity* theorems are applied in the constructions. One reason for investigating the separability of infinite sets of terms is the fact that there exist many countable sets of terms (numerals, sequences) that have to be discriminated inside the λ–calculus. Another motivation (stated by Böhm and others) is the connection between the separability and solvability of systems of equations in λ-calculus (see for example [7]).

A *discriminator* for a term $M \in \mathcal{M}$ is a term D_M such that for all $N \in \mathcal{M}$

$$D_M N = \begin{cases} \mathsf{T}, & \text{if} \quad N = M, \\ \mathsf{F}, & \text{otherwise,} \end{cases}$$

where: $\mathsf{T} = \lambda xy.x$ and $\mathsf{F} = \lambda xy.y$ are the terms representing true and false in λ–calculus. We shall say a term $M \in \mathcal{M}$ is *discriminable* in $\mathcal{M}$ if it has a discriminator in $\mathcal{M}$. The set of terms $\mathcal{M}$ has discriminators, or $\mathcal{M}$ is discriminable if each of its members has a discriminator. A stronger property is complete discriminability: $\mathcal{M}$ is *completely discriminable* if there exists a term D such that

$$DMN = \begin{cases} \mathsf{T}, & \text{if} \quad N = M, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

The term D is called a *complete discriminator* for $\mathcal{M}$.

A finite set of closed terms $\{M_1, \ldots, M_n\}$ is *separable* if there exists a term F such that $FM_i = x_i$, where x_i are pointwise distinct variables. A finite set of terms is separable if the set of their closures is separable. Hence the set of separable terms (resp. closed separable terms) can be mapped by an appropriate context (resp. term) onto any set of terms of less or equal cardinality. This notion can be generalized to infinite sets. An infinite set of

This paper is in final form and no version of it will be submitted for publication elsewhere.

terms is separable if there exists a term mapping the closures of its members one–to–one to the set of numerals (and hence to any sequence). An *adequate numeral system* is a countable set $\mathcal{N} = \{\ulcorner 0\urcorner, \ulcorner 1\urcorner, \ldots, \ulcorner n\urcorner, \ldots\}$ of terms (usually but not necessarily normal forms) associated with three terms:

1. the *test of zero* **Zero** such that $\mathbf{Zero}\ \ulcorner 0\urcorner = \mathsf{T}$ and $\mathbf{Zero}\ \ulcorner n\urcorner = \mathsf{F}$ for $n = 1, 2, \ldots,$
2. the *successor* R^+ such that $R^+ \ulcorner n\urcorner = \ulcorner n+1\urcorner$,
3. the *predecessor* R^- such that $R^- \ulcorner n+1\urcorner = \ulcorner n\urcorner$.

We shall call an adequate numeral system simply a *numeral system.*

The implications as follows are evident for the closed terms:

$\mathcal{M}$ is a numeral system
$\Rightarrow$ $\mathcal{M}$ is separable
$\Rightarrow$ $\mathcal{M}$ is completely discriminable
$\Rightarrow$ $\mathcal{M}$ is discriminable.

The preceding implications are proper.

A *sequence* of terms is a set of terms defined *uniformly* as follows. $\{M_n | M_n = F\ulcorner n\urcorner, n \in \mathbb{N}\}$, where F is an appropriate term – a *generator* of the sequence.

Recursive definitions of terms are based on 'paradoxical combinators'. We shall use the term $\Theta \equiv (\lambda xy.y(xxy))(\lambda xy.y(xxy))$, a well-known fixed point combinator, for this purpose. A characteristic example is as follows. A general recursive function f defined by:

$$f(n) = \begin{cases} g, & \text{if} \quad n = 0, \\ h(f(n-1)), & \text{if} \quad n \neq 0, \end{cases}$$

can be represented by a term F such that:

$$F\ulcorner n\urcorner = \mathbf{Zero}\ulcorner n\urcorner\ G(HF(R^-\ulcorner n\urcorner)),$$

where G, H represent g and h, respectively. A solution of this equation is

$$F = \Theta(\lambda xy.\mathbf{Zero}\, y\, G\, (Hx(R^- y))).$$

The sets of all λ–terms, of λ–terms with the free variables in $\{\vec{x}\}$ and of closed λ–terms, will be denoted as usual Λ, $\Lambda(\vec{x})$ and Λ^0, respectively. Let us choose an effective enumeration of the set $\Lambda(\vec{x})$ and denote $\natural M$ the number assigned to the term M. Hence the numeral of $\natural M$ in a fixed numeral system is $\ulcorner \natural M\urcorner$. We shall write $\ulcorner M\urcorner$ instead of $\ulcorner \natural M\urcorner$ since it is clear from the context in which sense the symbol $\ulcorner\ \urcorner$ is used (if we remember that a fixed enumeration of objects, i.e. λ-terms, and a fixed numeral system is supposed.) A term E so that for all $M \in \Lambda(\vec{x})$, $E\ulcorner M\urcorner = M$ is available (see Barendregt [1] 8.1.7.).

Effective functions on objects (nodes,finite sets, trees) will be considered as computable functions on the numbers corresponding to objects and (Church thesis is supposed) hence partially recursive. Since in λ–calculus all partial recursive functions are representable, the construction of numerical algorithms will be enough to prove the existence of a term with the desired properties when applied on numerals. This fact and Church thesis will be used without mentioning them.

It is well–known that $M \in \Lambda$ either is unsolvable or it has a *head normal form* (shortly *hnf*) $\lambda x_1 \ldots x_r.yM_1 \ldots M_s$. If M reduces to the hnf $\lambda x_1 \ldots x_r.yM_1 \ldots M_s$ via *head reductions* only $\lambda x_1 \ldots x_r.yM_1 \ldots M_s$ is called the *principal hnf* of M. The *Böhm tree* of $M \in \Lambda$ is defined recursively as follows:

$$BT(M) = \begin{cases} \bot, & \text{if M is unsolvable,} \\ \begin{array}{c} \lambda x_1 \ldots x_r.y \\ / \quad \cdots \quad \backslash \\ BT(M_1) \ldots BT(M_s) \end{array} & \begin{array}{l} \text{if } \lambda x_1 \ldots x_r.yM_1 \ldots M_s \\ \text{is the principal } hnf \text{ of } M. \end{array} \end{cases}$$

If $BT(M)$ is pruned at the depth k, $BT^k(M)$ will denote the resulting tree. Nodes of Böhm trees will be denoted as usually by finite sequences of numbers. For $\alpha \in Seq$ we shall denote $lh(\alpha)$ the length of α. $M|_\alpha$ will denote the label of the Böhm tree of the term M at the node α. Let us suppose the Böm tree is developed only until the depth $d = lh(\alpha)$, i.e. head reductions have been executed until the depth d. The term that "seats" at the node α after this operation will be called subterm at (the node) α and denoted M_α. The term M is *equivalent at* (the node) α to N ($M \sim_\alpha N$) if M_α and N_α are both unsolvable or $M_\alpha = \lambda x_1 \ldots x_m.yM_1 \ldots M_n$ and $N_\alpha = \lambda x_1 \ldots x_{m'}.zN_1 \ldots N_{n'}$ and $m - n = m' - n'$ and $y = z$ (more precisely y and z are both bounded at i–th place at a node β where $lh(\beta) \leq lh(\alpha)$ (i.e. $y = z = t_i$ for some t_i) or y and z are the same free variable in M_α or N_α. $\sim_{\langle\rangle}$ will be written $\sim$.

2. The Böhm–out technique

The Böhm–out technique (first used in [3]) is a method of extracting an appropriate subterm, in fact a substitutional instance of the head normal form at a given node α from the Böhm tree of a term. This is made using Böhm transformations ([1] p.p. 246–249). We shall modify these transformations so that an appropriate class of infinite sets of terms can be treated. The problem which arises is that the number of "fresh" variables used in the construction of the Böhm transformations is not limited. Our idea is to replace the fresh variables in Böhm transformations by another appropriate terms which can be recursively defined. We shall use for this purpose terms of the form $a\ulcorner i \urcorner$ where $i \in \mathbb{N}$ and a is a fresh variable. The result is a sharpening of the Böhm-out technique; it is made in some sense uniform on an appropriate subset of Λ which is denoted $\Lambda_f^T(\vec{x})$.

Let f be any total recursive function of one variable. Then for all natural x there exist natural m and n such that $\langle m,n\rangle = f(x)$, where $\langle\,.\,,\,.\,\rangle$ is a surjective recursive pairing. Hence the projections fst and snd are total recursive functions so that $fst \circ \langle m,n\rangle = m$ and $snd \circ \langle m,n\rangle = n$. In the next we shall use the notions of M_α and of the *hnf* of M as defined in the introduction.

Definition 2.1. *1. Let f be a total recursive function. The set of terms $\Lambda_f(\vec{x}) \subset \Lambda$ is defined as follows. $M \in \Lambda_f(\vec{x})$ if for all $\alpha \in Seq$, M_α is either unsolvable or it has a hnf $\lambda y_1 \ldots y_r.zM_1 \ldots M_s$, such that $0 \le r \le m_\alpha, 0 \le s \le n_\alpha$, where $\langle m_\alpha, n_\alpha\rangle = f(lh(\alpha))$.*

2. Let T be a tree. $M \in \Lambda_f^T(\vec{x}) \subset \Lambda$ if the condition for M_α given in (1) is valid for all $\alpha \in nodes(T)$.

We shall omit the subscripts α whenever it is possible. The number r (resp. s) is called the *order* (resp. the *degree*) of M_α, hence $M \in \Lambda_f(\vec{x})$ if and only if the order and the degree of any M_α are bounded by a total recursive function. Clearly, either $z \in \{\vec{x}\} = \{x_1,\ldots,x_l\}$ or it is bounded at a depth less than or equal $lh(\alpha)$. Using this terminology the preceding definition can be retold as follows: $\Lambda_f(\vec{x})$ is the set of terms such that each their subterm has the *order* at most $m = fst \circ f(lh(\alpha))$, the *degree* at most $n = snd \circ f(lh(\alpha))$ and all free variables are from the set $\{x_1,\ldots,x_l\}$. The item (2) of the definition is a relativization, i.e. the conditions in the definition are valid for subterms whose nodes lie in T. Evidently $\Lambda_f^T(\vec{x}) \ne \Lambda$ iff $T \ne \emptyset$. The use of the symbols r, m, m_α, s, n, n_α and l as defined in this paragraph will be standard throughout this section.

The following properties are evident for a set $\Lambda_f^T(\vec{x})$.

$$\begin{array}{ll} (1) & f \le g \Rightarrow \Lambda_f^T(\vec{x}) \subseteq \Lambda_g^T(\vec{x}), \\ (2) & T_1 \subseteq T_2 \Rightarrow \Lambda_f^{T_2}(\vec{x}) \subseteq \Lambda_f^{T_1}(\vec{x}), \\ (3) & \{\vec{x}\} \subseteq \{\vec{y}\} \Rightarrow \Lambda_f^T(\vec{x}) \subseteq \Lambda_f^T(\vec{y}). \end{array} \quad (2.1)$$

A *hnf* $\lambda x_1 \ldots x_r.yM_1 \ldots M_s$ is called *λ–free* if r=0 and *head original* if $y \notin FV(\vec{M})$. A term is *ready* if it is unsolvable or has a λ–free and head original *hnf*.

Before we begin explaining our construction, let us recall some known notions and results we shall need (see Barendregt [1] pp.245-254 for the details).

A permutator P_n^σ (where σ is a permutation) is a term permuting its arguments accordingly to σ, when it is applied to $n+1$ terms $M_1 \ldots M_{n+1}$: i.e. $P_n^\sigma M_1 \ldots M_{n+1} = M_{\sigma(1)} \ldots M_{\sigma(n+1)}$. We shall need only a special kind of permutators, i.e. $P_n \equiv \lambda x_1 \ldots x_{n+1}.x_{n+1}x_1 \ldots x_n$, and we shall call them *permutators.* The next evident property of permutators are crucial in our construction.

Fact 2.2. *For $m \ge n+1$*

$$P_nM_1 \ldots M_m = M_{n+1}M_1 \ldots M_nM_{n+2} \ldots M_m.$$

A *selector* is a term $U_i^n \equiv \lambda x_0 \ldots x_n.x_i$, $0 \le i \le n$. First we shall prove that the permutators $P_n \equiv \lambda x_1 \ldots x_{n+1}.x_{n+1}x_1 \ldots x_n$, and the selectors $U_i^n \equiv \lambda x_0 \ldots x_n.x_i$, $0 \le i \le n$ can be recursively defined.

Lemma 2.3. *There exist terms P and U such that*

$$\begin{array}{ll} (1) & P_n = P\ulcorner n \urcorner; \\ (2) & U_i^n = U\ulcorner n \urcorner \ulcorner i \urcorner. \end{array}$$

Proof.

1. Let us define

$$Q_n = \lambda y x_1 \ldots x_{n+1}. y x_{n+1} x_1 \ldots x_n$$

and

$$A \equiv \lambda xyz.x(\lambda t.ytz).$$

Then one has: $AQ_n = Q_{n+1}$ and $P_n = Q_n I$ where $I = \lambda x.x$ is the identity combinator.

Let Q be recursively defined as follows.

$$Q = \Theta(\lambda xy.\, \mathbf{Zero}\, y\, Q_0\, (A(x(R^- y)))).$$

It is easy to verify that $Q\ulcorner n \urcorner = Q_n$. Therefore we can define

$$P = \lambda z.QzI.$$

2. Let $K = \top = \lambda xy.x$ and $B = \lambda xyz.xy$. Then $KU_i^n = U_{i+1}^{n+1}$, $BU_0^n = U_0^{n+1}$, and finally $U_i^n = K^i(B^{n-i}U_0^0)$. Now the construction of U is straightforward, by applying twice the fixed point combinator:

Let us construct a term Dif so that $Dif\ulcorner m \urcorner \ulcorner n \urcorner = \ulcorner m-n \urcorner$ for $m \geq n$. Dif accomplishes the following equation:

$$Dif\ulcorner m \urcorner \ulcorner n \urcorner$$
$$= \mathbf{Zero}\ulcorner n \urcorner \ulcorner m \urcorner (R^+(Dif\,(R^- \ulcorner m \urcorner)\ulcorner n \urcorner)).$$

Hence

$$Dif \equiv \Theta(\lambda xyz.\mathbf{Zero}\, zy(R^+(x(R^- y)z))).$$

For any term F $F^n M$ can be recursively defined by the aid of the term

$$\hat{F} \equiv \Theta(\lambda xyz.\mathbf{Zero}\, yI(F(x(R^- y)z))),$$

since

$$\hat{F}\ulcorner m \urcorner M = \mathbf{Zero}\ulcorner m \urcorner I(F(\hat{F}(R^- \ulcorner m \urcorner)M)),$$

hence $F^n M = \hat{F}\ulcorner n \urcorner M$. Using the notation from the preceding equations, U is given explicitly as follows:

$$U \equiv \lambda xy.\hat{K}\, y(\hat{B}(Dif\, x\, y)U_0^0),$$

where K is as usual.

□

Remark 2.4. The referee suggested that using Church numerals we can avoid the use of Θ and we obtain results in normal forms.

Let c_n denotes $\ulcorner n \urcorner$ in Church representation, i.e. $c_n = \lambda f x.f^n x$. We have that $Q = \lambda x.xAQ_0$ satisfies $Qc_n = Q_n$. Analogously

$$U = \lambda xy.yK(yR^- xBU_0^0)$$

satisfies item (2) of the preceeding lemma.

Let $M \in \Lambda_f^T(\vec{x})$, where $T \neq \emptyset$ s.t. M is unsolvable or $M = \lambda y_1 \ldots y_r.zM_1 \ldots M_s$, where $0 \leq r \leq m = m_{\langle\rangle} = fst \circ f(0)$ and $0 \leq s \leq n = n_{\langle\rangle} = snd \circ f(0)$. Let a be a fresh variable (that is $a \notin \{x_1, \ldots, x_l\} = FV(\Lambda(\vec{x}))$). Let us define

$$F_{1,m} \equiv \lambda t.t(a\ulcorner 1 \urcorner)\ldots(a\ulcorner m \urcorner). \tag{2.2}$$

For any term N N^* will denote a substitutional instance of N, i.e. $N^* = N[\vec{y} := \vec{P}]$ for some $\vec{y}$ and $\vec{P}$. Then

$$F_{1,m}M = \begin{cases} x_i M_1^* \ldots M_s^*(a\ulcorner r+1 \urcorner)\ldots(a\ulcorner m \urcorner), \\ \qquad \text{if} \quad z \equiv x_i, \\ a\ulcorner j \urcorner M_1^* \ldots M_s^*(a\ulcorner r+1 \urcorner)\ldots(a\ulcorner m \urcorner), \\ \qquad \text{if} \quad z \equiv y_j, \\ \text{unsolvable}, \\ \qquad \text{if M is unsolvable}, \end{cases} \tag{2.3}$$

for any $x_i \in \{x_1, \ldots, x_l\}$ or any $0 \leq j \leq r$, i.e. $y_j \in \{y_1, \ldots, y_r\}$.

Let k be a natural number and b be a fresh variable. Let

$$F_{2,m}^k \equiv \lambda t.F_{1,m}t(b\ulcorner 1 \urcorner)\ldots(b\ulcorner k \urcorner), \tag{2.4}$$

hence $F_{2,m}^k M = F_{1,m}M(b\ulcorner 1 \urcorner)\ldots(b\ulcorner k \urcorner)$.

Let us define a transformation ρ on terms as follows.

$$M^\rho = (F_{2,m}^k M)[x_1 := a\ulcorner m+1 \urcorner]\ldots[x_l := a\ulcorner m+l \urcorner].$$

Explicitly:

$$M^\rho = a\ulcorner i \urcorner M_1^* \ldots M_s^*(a\ulcorner r+1 \urcorner)\ldots (a\ulcorner m \urcorner)(b\ulcorner 1 \urcorner)\ldots(b\ulcorner k \urcorner), \tag{2.5}$$

where $1 \leq i \leq m+l$, when M has a *hnf*, and otherwise ρ transforms an unsolvable term into an unsolvable term.

Let $dg(M)$ denotes the degree of M.

Fact 2.5. *1. The numerals after the head variable of M^ρ are in one-to-one correspondence with the head variables of M,*

2. $dg(M^\rho)$ is $m+k-r+s+1$,

3. M is a closed term implies $M^\rho = F^k_{2,m}M$.

Proof.

1. By equations 2.3 and 2.5 it is clear that j corresponds to a bound head variable y_j if $1 \leq j \leq m$ and corresponds to a free variable x_{j-m} if $m < j \leq m+l$.
2. By equation 2.5. Since $j \leq r$ or $j > m$ the head variable a is followed by $\ulcorner i \urcorner$, s M^*'s, $m-r$ $a\ulcorner u \urcorner$'s and k $b\ulcorner v \urcorner$'s (see Example 2.6.)
3. Immediately by the definition of M^ρ.

□

Let us enumerate the $\sim$ −equivalence classes of $\Lambda^T_f(\vec{x})$. For any head variable $x_i \in \{x_1, \dots x_l\}$ the inequality $-n \leq r-s \leq m$ is accomplished. Hence there exist $l(m+n+1)$ possibilities for the difference $r-s$. Similarly for the bound head variable y_i there exist $m+n+1-i$ possibilities for the difference $r-s$. We shall denote $p = l(m+n+1) + (m+n+1-1) + \dots + (m+n+1-i) + \dots + (n+1) = (l+m)(m+n+1) - m(m+1)/2$ the number of $\sim$-equivalence classes of terms of $\Lambda^T_f(\vec{x})$ having a *hnf* (there is one more class of unsolvable terms).

The idea is now to replace the head terms $a\ulcorner i \urcorner$ by terms $b\ulcorner j \urcorner$ so that j's are in one-to-one correspondence with the $\sim$ −equivalence classes. Of course it must be supposed $k \geq p$. This replacement will be done with the aid of an appropriate permutator which will pick up a $b\ulcorner j \urcorner$ and will put it at the head of M^ρ. The following example illustrates the idea.

Example 2.6. *Let*

1. *$f(x) = \langle 2, 2 \rangle$, where $x = lh(\alpha)$, $\alpha \in Seq$, hence the order m_α and the degree n_α of any M_α are both equal 2 for a $M \in \Lambda_f(\vec{x})$,*
2. *$FV(\Lambda_f(\vec{x})) = \{x_1\}$, i.e. $l = 1$.*

The number of $\sim$ −equivalence classes p is 12. Applications of $F_{1,m}$ and $F^k_{2,m}$ (k=12) to the term $M = \lambda y_1.x_1M_1$, and M^ρ are respectively as follows.

$$\begin{aligned} F_{1,m}M &= x_1M_1^*(a\ulcorner 2 \urcorner), \\ F^k_{2,m}M &= x_1M_1^*(a\ulcorner 2 \urcorner)(b\ulcorner 1 \urcorner)\dots(b\ulcorner 12 \urcorner), \\ M^\rho &= a\ulcorner 3 \urcorner M_1^*(a\ulcorner 2 \urcorner)(b\ulcorner 1 \urcorner)\dots(b\ulcorner 12 \urcorner). \end{aligned}$$

Let us consider the $\sim$ −equivalence classes of $\Lambda_f(\vec{x})$. Four of them correspond to the head variable y_1, i.e. head variable bound by the leftmost lambda abstraction in the hnf of the term (say N). Note that this means that the first two terms of N^ρ are $a\ulcorner 1 \urcorner$. Analogously three $\sim$ −equivalence classes correspond to $a\ulcorner 2 \urcorner$, and five to $a\ulcorner 3 \urcorner$.

In Figure 1 all possible hnf's of terms from $\Lambda_f(\vec{x})$ and their ρ−transforms are collected. In the third column of the Figure 1 there are equivalence class numbers $eqn(M^\rho)$ (see Definition 2.8 (2) computed by the algorithm in the proof of Proposition 2.9. In the fourth column there are the degrees of the terms M^ρ. Note that $eqn(M^\rho)$'s which correspond to a given head variable lie on an interval. Hence the numeral i succiding the head variable a in M^ρ can be computed from the given $j = eqn(M^\rho)$. Let us denote j^* the maximal j corresponding to the same i as j. Note $j^* = eqn((\lambda y_1 \dots y_m.y_i)^\rho)$ or $j^* = eqn((x_i)^\rho)$. Hence j^* can be computed from i, hence from j. Hence the degree of M^ρ can be computed from j ($dg(M^\rho) = 13 + j^* - j$ in the present example).

For internalizing the previously mentioned substitution the fact that $F_{1,m}$ and $F^k_{2,m}$ can be defined recursively, is required.

Lemma 2.7. *There exist terms F_1 and F_2 such that*

$$F_1\ulcorner m \urcorner = F_{1,m}, \qquad F_2\ulcorner m \urcorner \ulcorner k \urcorner = F^k_{2,m}.$$

Proof. We shall prove

$$\begin{aligned} F_1 &\equiv \Theta(\lambda xy.\mathbf{Zero}y I(\lambda u.x(R^- y)u(ay))), \\ F_2 &\equiv \Theta(\lambda xyz.\mathbf{Zero}z(F_1 y)(\lambda u.xy(R^- z)u(bz))). \end{aligned}$$

Note

$$\begin{aligned} F_1\ulcorner m \urcorner &= \mathbf{Zero}\ulcorner m \urcorner I(\lambda u.F_1(R^-\ulcorner m \urcorner)u(a\ulcorner m \urcorner)), \\ F_2\ulcorner m \urcorner \ulcorner k \urcorner &= \mathbf{Zero}\ulcorner k \urcorner (F_1\ulcorner m \urcorner) \\ &\quad (\lambda u.F_2\ulcorner m \urcorner(R^-\ulcorner k \urcorner)u(b\ulcorner k \urcorner)). \end{aligned}$$

Class		M^ρ	eqn	dg
1.	$(\lambda y_1 y_2.y_1)^\rho$	$= a\ulcorner 1\urcorner(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	4	13
2.	$(\lambda y_1.y_1)^\rho$	$= a\ulcorner 1\urcorner(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	3	14
	$(\lambda y_1 y_2.y_1 M_1)^\rho$	$= a\ulcorner 1\urcorner M_1^*(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$		
3.	$(\lambda y_1.y_1 M_1)^\rho$	$= a\ulcorner 1\urcorner M_1^*(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	2	15
	$(\lambda y_1 y_2.y_1 M_1 M_2)^\rho$	$= a\ulcorner 1\urcorner M_1^* M_2^*(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$		
4.	$(\lambda y_1.y_1 M_1 M_2)^\rho$	$= a\ulcorner 1\urcorner M_1^* M_2^*(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	1	16
5.	$(\lambda y_1 y_2.y_2)^\rho$	$= a\ulcorner 2\urcorner(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	7	13
6.	$(\lambda y_1 y_2.y_2 M_1)^\rho$	$= a\ulcorner 2\urcorner M_1^*(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	6	14
7.	$(\lambda y_1 y_2.y_2 M_1 M_2)^\rho$	$= a\ulcorner 2\urcorner M_1^* M_2^*(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	5	15
8.	$(\lambda y_1 y_2.x_1)^\rho$	$= a\ulcorner 3\urcorner(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	12	13
9.	$(\lambda y_1.x_1)^\rho$	$= a\ulcorner 3\urcorner(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	11	14
	$(\lambda y_1 y_2.x_1 M_1)^\rho$	$= a\ulcorner 3\urcorner M_1^*(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$		
10.	$(x_1)^\rho$	$= a\ulcorner 3\urcorner(a\ulcorner 1\urcorner)(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	10	15
	$(\lambda y_1.x_1 M_1)^\rho$	$= a\ulcorner 3\urcorner M_1^*(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$		
	$(\lambda y_1 y_2.x_1 M_1 M_2)^\rho$	$= a\ulcorner 3\urcorner M_1^* M_2^*(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$		
11.	$(x_1 M_1)^\rho$	$= a\ulcorner 3\urcorner M_1^*(a\ulcorner 1\urcorner)(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	9	16
	$(\lambda y_1.x_1 M_1 M_2)^\rho$	$= a\ulcorner 3\urcorner M_1^* M_2^*(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$		
12.	$(x_1 M_1 M_2)^\rho$	$= a\ulcorner 3\urcorner M_1^* M_2^*(a\ulcorner 1\urcorner)(a\ulcorner 2\urcorner)(b\ulcorner 1\urcorner)\dots(b\ulcorner 12\urcorner)$	8	17

Fig. 1. ρ–transforms of *hnf s* having the order at most 2 and the degree at most 2.

The proof is by the induction on m resp. k.

Note that $F_1\ulcorner 0\urcorner = I = F_{1,0}$. Let $M > 0$.

$$\begin{aligned} F_1\ulcorner m\urcorner &= \lambda u.F_1(R^-\ulcorner m\urcorner)u(a\ulcorner m\urcorner) \\ &= \lambda u.(\lambda t.t(a\ulcorner 1\urcorner)\dots(a\ulcorner m-1\urcorner)) \\ &\quad u(a\ulcorner m\urcorner) \qquad \text{induction hypothesis} \\ &= \lambda u.u(a\ulcorner 1\urcorner)\dots(a\ulcorner m\urcorner). \end{aligned}$$

Analogously $F_2\ulcorner m\urcorner\ulcorner 0\urcorner = \lambda t.F_{1,m}t = F_{1,m}$. Let $k > 0$.

$$\begin{aligned} F_2\ulcorner m\urcorner\ulcorner k\urcorner &= \lambda u.F_2\ulcorner m\urcorner(R^-\ulcorner k\urcorner)u(b\ulcorner k\urcorner) \\ &= \lambda u.(\lambda t.F_{1,m}t(b\ulcorner 1\urcorner)\dots(b\ulcorner k-1\urcorner)) \\ &\quad u(b\ulcorner k\urcorner) \qquad \text{induction hypothesis} \\ &= \lambda u.F_{1,m}u(b\ulcorner 1\urcorner)\dots(b\ulcorner k\urcorner). \end{aligned}$$

□

The next step is the replacement of $a\ulcorner i\urcorner$ in M^ρ by a $b\ulcorner j\urcorner$ such that j reflects the $\sim$ –equivalence class of M.

Definition 2.8. *1. A transformation $\pi : \Lambda_f^T(\vec{x}) \to \Lambda$ is faithful if*

(a) for any solvable $M, N \in \Lambda_f^T(\vec{x})$ M^π has the form

$$b\ulcorner j\urcorner M_1^* \dots M_s^*(a\ulcorner r+1\urcorner \dots \dots (a\ulcorner m\urcorner)(b\ulcorner 1\urcorner)\dots \dots (b\ulcorner j-1\urcorner)(b\ulcorner j+1\urcorner)\dots(b\ulcorner k\urcorner),$$

and

$$M \sim N \quad \text{iff} \quad M^\pi = b\ulcorner j\urcorner M_1^* \dots \quad \text{and} \quad N^\pi = b\ulcorner j\urcorner N_1^* \dots,$$

i.e. the numerals lying after the head variable b are in one-to-one correspondence with $\sim$ –equivalence classes,

(b) M^π is unsolvable if and only if M is unsolvable.

2. A number j is the $\sim$ –equivalence class number of the term $M \in \Lambda_f^T(\vec{x})$ with respect to the faithful transformation π if $M^\pi = b\ulcorner j\urcorner M_1^* \dots$

The equivalence class number of the term M will be denoted $eqn(M)$.

Proposition 2.9. *$T \neq \emptyset$ implies*

1. *there exists a faithful transformation $\pi : \Lambda_f^T(\vec{x}) \to \Lambda$,*
2. *there exists a term Π such that for any closed term $M \in \Lambda_f^T(\vec{x})$ $\Pi M = M^\pi$,*
3. *For any M_i^* $b \notin FV(M_i^*)$.*

Proof.

1. We shall construct an algorithm that will replace the head terms $a\ulcorner i \urcorner$ by $b\ulcorner j \urcorner$ where $\ulcorner j \urcorner$ depends on the $\sim$ −equivalence classes of M only.

   Let us define a function φ as follows.

$$\varphi(i) = \begin{cases} i(n{+}1), & \text{if } m = 0, \\ (m{+}n){+}(m{+}n{-}1) \\ \quad +\ldots+(m{+}n{-}(i{-}1)){-}1; & \text{if } 1{\leq}i{\leq}m, \\ \varphi(m){+}(i{-}m)(m{+}n{+}1); & \text{if } m{\leq}i{\leq}m+l. \end{cases}$$

   The announced algorithm is as follows: pick up a $b\ulcorner j \urcorner$ and replace $a\ulcorner i \urcorner$ by it, i.e.

$$M^\rho[a\ulcorner i \urcorner := P_{\varphi(i)}] = b\ulcorner j \urcorner M_1^* \ldots$$

   The function φ is chosen so that supposing $k \geq p$, different j's correspond to different $\sim$ −equivalent classes of solvable terms from $\Lambda_f^T(\vec{x})$. This is easy to see on Example 2.6, but the general case is treated analogously:

$$\begin{array}{lll} i = 1 & \varphi(1) = 3 & 1 \leq j \leq 4, \\ i = 2 & \varphi(2) = 6 & 5 \leq j \leq 7, \\ i = 3 & \varphi(3) = 11 & 8 \leq j \leq 12. \end{array}$$

   The next step is an internalization of the preceding algorithm. Let Φ be a term representing the function φ in λ−calculus. 2.3 implies $P_{\varphi(i)} = P(\Phi\ulcorner i \urcorner)$. Hence the transformation π can be defined as follows.

$$M^\pi = (\lambda a.M^\rho)(\lambda x.(P(\Phi x))).$$

   Notice that with the obvious choice for P we will have $dg(M^\pi) = dg(M^\rho) - 1$.

2. Since the substitution of free variables $x_1, \ldots, x_l$ is the only action which cannot be internalized in λ−calculus, i.e. substituted by an application, the assertion is immediately by 2.5 (3) and (1).

3. Evidently by the construction of π.

□

Let us suppose f and g are general recursive functions so that $fst \circ f \leq fst \circ g$ and $snd \circ f \leq snd \circ g$. Then $\Lambda_f^T(\vec{x})$ is a subset of $\Lambda_g^T(\vec{x})$ for any tree T, by the equation 2.1 (1). Hence

Fact 2.10. *$\pi : \Lambda_g^T(\vec{x}) \to \Lambda$ is a faithful transformation implies the restriction of π to $\Lambda_f^T(\vec{x})$ is a faithful transformation.*

Remark 2.11. 1. Note that the equivalence class number with respect to the algorithm represented by φ contains a complete information about the $\sim$ −equivalence class, in particular the information about the length of M^π, while the head $a\ulcorner i \urcorner$ of the term M^ρ contains only information about the head variable. Despite of the lack of information the construction was successful since only permutators were used and so the exact length of the term was not necessary. The situation is strictly different when a selector is applied to a term; the length must be exactly known because to choose the right selector.

2. The assertion (3) of the preceding proposition is an analogon to the head originality in the finite case (cf. [1] p.247).

Proposition 2.12. *Let M be any member of $\Lambda_f^T(\vec{x})$ where $T \neq \emptyset$, having a hnf $\lambda x_1 \ldots x_r.zM_1 \ldots M_s$. Then*

1. *there exists a transformation η depending on the parameter i such that for any $1 \leq i \leq s$ $M^{\eta(i)} = M_i^*$,*
2. *there exists a transformation ν such that $M^\nu = \ulcorner eqn(M) \urcorner$,*
3. *there exist terms Ext and Eqn such that if $M \in \Lambda_f^T(\vec{x})$ is closed, then*

$$Ext\ulcorner i \urcorner M = M^{\eta(i)} \qquad Eqn\, M = M^\nu.$$

Proof.

1. Let us suppose $k = p$, hence Fact 2.5 (2) implies $dg(M^{\rho})$, where $M \in \Lambda_f^T(\vec{x})$ depends on m, n, and $r - s$. Remember the terms having different $r - s$'s lie in the different $\sim$ – equivalence classes. Hence $dg(M^{\rho})$ depends on the $\sim$ – equivalence class of M. Let us suppose the transformation π is fixed, hence the correspondence between the $\sim$ – equivalence classes of $\Lambda_f^T(\vec{x})$ and $eqn(M)$'s, i.e. $\ulcorner j \urcorner$'s (the numerals succeeding the head variable b in M^{π}, $M \in \Lambda_f^T(\vec{x})$) is fixed. Hence $dg(M^{\rho})$ can be computed from m, n, and $eqn(M)$ (see the proof of 2.9 (1) and 2.6 for an example of such computation). Hence $dg(M^{\pi}) = dg(M^{\rho}) - 1$ can be computed from m, n, and $eqn(M)$. Let us denote Lh a term such that $Lh\ulcorner m \urcorner \ulcorner n \urcorner \ulcorner eqn(M) \urcorner = \ulcorner dg(M^{\pi}) - 1 \urcorner$ (that is the degree of the term without the head term $\ulcorner j \urcorner$). The transformation η is as follows.

   $$M^{\eta(i)} = (\lambda b.M^{\pi})(\lambda x.U\,(R^{-}(Lh\ulcorner m \urcorner \ulcorner n \urcorner x))(R^{-}\ulcorner i \urcorner)),$$

   where U is as in 2.3. The correctness of the preceding definition is shown by a simple calculation on an example, but the general case is treated analogously. Let $m = n = 2$, $l = 1$, $M \equiv \lambda y_1.x_1 M_1 M_2$ (see Example 2.6, Figure 1 and the proof of 2.9). Hence

   $$M^{\pi} = b\ulcorner 9 \urcorner M_1^* M_2^* (a\ulcorner 2 \urcorner)(b\ulcorner 1 \urcorner)\ldots (b\ulcorner 8 \urcorner)(b\ulcorner 10 \urcorner)\ldots(b\ulcorner 12 \urcorner).$$

   Remember $dg(M^{\pi}) = dg(M^{\rho}) - 1 = 15$. Let us suppose $i = 2$.

   $$\begin{aligned} M^{\eta(i)} &= (\lambda b.b\ulcorner j \urcorner M_1^* \ldots) \\ &\quad (\lambda x.U\,(R^{-}(Lh\ulcorner m \urcorner \ulcorner n \urcorner x))(R^{-}\ulcorner i \urcorner)) \\ &= U(R^{-}(Lh\ulcorner m \urcorner \ulcorner n \urcorner \ulcorner j \urcorner))(R^{-}\ulcorner i \urcorner)M_1^* \ldots \\ &= U_{i-1}^{dg(M^{\pi})-2} M_1^* \ldots \\ &= U_1^{13} M_1^* M_2^* (a\ulcorner 2 \urcorner)(b\ulcorner 1 \urcorner)\ldots \\ &\quad (b\ulcorner 8 \urcorner)(b\ulcorner 10 \urcorner)\ldots(b\ulcorner 12 \urcorner) \\ &= M_2^*. \end{aligned}$$

2. The transformation ν is defined as follows:

   $$M^{\nu} = (\lambda b.M^{\pi})(\lambda x.U\,(Lh\ulcorner m \urcorner \ulcorner n \urcorner x)\ulcorner 0 \urcorner x).$$

   Let us use the same example as in (1) and calculate:

   $$\begin{aligned} M^{\nu} &= (\lambda b.b\ulcorner 9 \urcorner M_1^* \ldots) \\ &\quad (\lambda x.U\,(Lh\ulcorner m \urcorner \ulcorner n \urcorner x)\ulcorner 0 \urcorner x) \\ &= U(Lh\ulcorner m \urcorner \ulcorner n \urcorner \ulcorner 9 \urcorner)\ulcorner 0 \urcorner \ulcorner j \urcorner M_1^* \ldots \\ &= U_0^{dg(M^{\pi})-1} \ulcorner 9 \urcorner M_1^* \ldots \\ &= U_0^{14} \ulcorner 9 \urcorner M_1^* M_2^* (a\ulcorner 2 \urcorner)(b\ulcorner 1 \urcorner)\ldots \\ &\quad (b\ulcorner 8 \urcorner)(b\ulcorner 10 \urcorner)\ldots(b\ulcorner 12 \urcorner) \\ &= \ulcorner 9 \urcorner. \end{aligned}$$

3. Immediately by the proofs of (1) and (2) and Fact 2.5 (3).

□

The next step is an iteration of the operation of extraction which will enable to reach a substitutional instance of a subterm at any node of the Böhm tree of the term.

Remember we denoted M_{α} ($\alpha \in Seq$) the *subterm at the node* α (of M), i.e. the subterm seating at the node α when on the term M only the head reduction to the depth $d = lh(\alpha)$ has been executed.

Lemma 2.13. *There exists a faithful transformation o such that for any term $N \in \Lambda_f^T(\vec{x})$, where $T \neq \emptyset$ and for any $\alpha \in nodes(T)$, if $lh(\alpha) = d$, $d \in \mathbb{N}$, then the subterm $N_{\alpha} = \lambda x_1 \ldots x_r.zN_{\alpha*\langle 1\rangle} \ldots N_{\alpha*\langle s\rangle}$ is transformed as follows.*

1. *If the head variable of N_{α} is bound at the root node or it is free then*

   $$\begin{aligned} (\lambda x_1 \ldots x_r.zN_{\alpha*\langle 1\rangle} \ldots N_{\alpha*\langle s\rangle})^o = \\ \lambda x_1 \ldots x_r y_{s+1} \ldots y_{\varphi(i)+1}.y_{\varphi(i)+1} \\ N^*_{\alpha*\langle 1\rangle} \ldots N^*_{\alpha*\langle s\rangle} y_{s+1} \ldots y_{\varphi(i)}, \end{aligned}$$

   where φ is the function defined in the proof of 2.9 and i is the number assigned to the head variable of M as in the equation 2.5.

2. *If the head variable is not as in the previous case, i.e. it is bound in a node different from the root then*

   $$\begin{aligned} &(\lambda x_1 \ldots x_r.zN_{\alpha*\langle 1\rangle} \ldots N_{\alpha*\langle s\rangle})^o \\ &= \lambda x_1 \ldots x_r.zN^*_{\alpha*\langle 1\rangle} \ldots N^*_{\alpha*\langle s\rangle}. \end{aligned}$$

Proof. The following modification of the transformation π of Proposition 2.9 is necessary to obtain o. Let $m = \max\{m_0, \ldots, m_d\}$, resp. $n = \max\{n_0, \ldots, n_d\}$, where $m_i = fst \circ f(i), n_i = snd \circ f(i)$ and $d = lh\alpha$ must be replaced to $m = m_0 = fst \circ f(0)$, resp. $n = n_0 = snd \circ f(0)$. Let us define $g = \langle m, n\rangle$ if $0 \leq i \leq d$ and $g = f$ if $i > d$. Then there exists a faithful transformation $\pi : \Lambda_g^T(\vec{x}) \to \Lambda$ by 2.9. Let o be the restriction of π to $\Lambda_f^T(\vec{x})$. Hence o is a faithful transformation by 2.10. The rest is a simple calculation as in [1] 10.3.6

□

Remark 2.14. Informally, if t in the permutator P_t is greater than or equal to the maximal degree of M_α, then an application of the transformation o to the term N doesn't mix the subtrees at any node α of $BT(N)$ lying in the tree T and having the length less or equal to d. Note that this condition is accomplished since $t = \varphi(i) \geq n \geq s$. Hence o doesn't mix the subtrees at the set of nodes $\{\alpha \in T | lh(\alpha) \leq d\}$. This fact together with 2.9 means that o doesn't lose any information about the set $\Lambda_f^T(\vec{x})$ to the depth d. In the finite case o coincides with the notion of *qualified* context. The notion of qualified context was introduced in [7] where an infinite number of such contexts was constructed.

Let $\eta(i)$ be the transformation extracting M_i^* as in 2.12 defined with respect to the transformation o from the 2.13. For any nonempty tree T and any $d \in \mathbb{N}$, T'_d will denote the subtree of T which will include all and only nodes having the depth less than or equal to d. As in previous lemma, g is the function obtained from f by replacing the $d+1$ initial first components of f by their maximal value and analogously the $d+1$ initial second components. Put $k = p = (l+m)(m+n+1) - m(m+1)/2$. Then k is the number of $\sim$ −equivalence classes of solvable terms of Λ_g^T. Let us denote h the general recursive function defined as follows.

$$fst \circ h = fst \circ f + k \qquad snd \circ h = k.$$

Corollary 2.15. *The transformation $\eta(i)$ increases the order of k and makes the degree equal k at any node having the length less or equal d for any term in $\Lambda_f^T(\vec{x})$, when applied to it, i.e.*

$$\eta(i) : \Lambda_f^{T'_d} \to (\Lambda_h^{T'_{d-1}})^0,$$

assuming $d > 0$.

Proof. $M^{\eta(i)}$ doesn't contain free variables x_j, nor a nor b since all of them were substituted by permutators or selectors. The orders and the degrees of the transformed subterms seating at the nodes of T' follow immediately by the preceding lemma since the depth of nodes in T' is less or equal d.

□

A generalization of the transformations η and ν to any node of the Böhm tree of the term is now straightforward by induction on the depth of the nodes. Let $g_0 = f$ and $g_{\nu+1} = h(g_\nu)$ where h is as in the previous Corollary 2.15. The algorithm for g_ν is clear: at each step of calculation the corresponding $m = \max\{fst \circ g_\nu(\nu+1), \ldots, fst \circ g_\nu(lh(\alpha))\}$ are taken, and analogously the other parameters n and k for computing $g_{\nu+1}$. Then the described method of extraction is repeated by using the new parameters.

Theorem 2.16. *[Generalized Böhm-out lemma]*

1. *There exists a transformation η depending on $\alpha \in Seq$ such that for any $M \in \Lambda_f^T(\vec{x})$ and any $\alpha \in nodes(BT(M)) \cap nodes(T)$*

$$M^{\eta(\alpha)} = \begin{cases} M_\alpha^*, & \text{if } M|_\alpha \downarrow, \\ \text{unsolvable} & \text{otherwise.} \end{cases}$$

2. *If M is a closed term then there exists a term B_0 so that*

$$B_0 \ulcorner\alpha\urcorner M \;= M^{\eta(\alpha)}.$$

Proof.

1. In the following we define some terms representing the general recursive functions we shall use in the construction. G is a term such that $G\ulcorner v\urcorner$ represents the function g_v, $G'_1\ulcorner v\urcorner$ (resp. $G'_2\ulcorner v\urcorner$) represents $fst \circ g_v$ (resp. $snd \circ g_v$), $G_1\ulcorner v\urcorner\ulcorner w\urcorner$ (resp. $G_2\ulcorner v\urcorner\ulcorner w\urcorner$) represents $\max\{fst \circ g_t | v \leq t \leq w\}$ (resp. $\max\{snd \circ g_t | v \leq t \leq w\}$). Suppose $v = lh(\beta) < w = lh(\alpha)$ and $\beta * i$ is a subsequence of α. By the induction hypothesis $\eta(\beta)$ is defined. Put

$\ulcorner m \urcorner = G_1 \ulcorner v \urcorner \ulcorner w \urcorner$ and $\ulcorner n \urcorner = G_2 \ulcorner v \urcorner \ulcorner w \urcorner$ in 2.12 (2.13 explains why m and n have been calculated as we stated) and define

$$M^{\eta(\beta * i)} = (M^{\eta(\beta)})^{\eta(i)}.$$

2. We shall prove the construction in (1) can be made inside the λ–calculus. Let
   - Z be a test for the non-emptiness of sequences, i.e. $Z \ulcorner \alpha \urcorner = \mathsf{T}$ if $\alpha \neq \langle\,\rangle$ and $Z \ulcorner \alpha \urcorner = \mathsf{F}$ if $\alpha = \langle\,\rangle$,
   - *Head* and *Tail* be (non standard) destructors of sequences, i.e. $\alpha = \beta * i$ implies $Head \ulcorner \alpha \urcorner = \ulcorner \beta \urcorner$ and $Tail \ulcorner \alpha \urcorner = \ulcorner i \urcorner$.

   A term computing the length of sequences is a solution of the equation:

   $$Lgh = \lambda x.\, Zx(R^+(Lgh(Head\, x)))\ulcorner 0 \urcorner.$$

   At each step $\ulcorner m \urcorner$ and $\ulcorner n \urcorner$ are calculated as in (1). Analogously as in the proof of 2.12 a term Lh is built such that

   $$Lh \ulcorner m \urcorner \ulcorner n \urcorner \ulcorner eqn(M) \urcorner = \ulcorner dg(M^{\pi}) - 1 \urcorner.$$

   The term B_0 is defined as follows.

   $$B_0 \ulcorner \alpha \urcorner M = Z \ulcorner \alpha \urcorner (Ext(Tail \ulcorner \alpha \urcorner)(B_0(Head \ulcorner \alpha \urcorner)M))M.$$

□

A generalization of the notion of $\sim$ –equivalence class number is straightforward as follows. The $\sim_\alpha$ *–equivalence class number* of the term M with respect to the transformation η is the $\sim$ –equivalence class number of $M^{\eta(\alpha)}$. $eqn_\alpha(M)$ will denote the $\sim_\alpha$ –equivalence class number of the term M.

The following corollary is an instance of the preceding theorem and of 2.12.

Corollary 2.17. *1. There exists a transformation ν depending on $\alpha \in Seq$ such that for any $M \in \Lambda_f^T(\vec{x})$ and any $\alpha \in nodes(BT(M)) \cap nodes(T)$*

$$M^{\nu(\alpha)} = \begin{cases} \ulcorner eqn_\alpha(M) \urcorner, & \text{if } M|_\alpha \downarrow, \\ unsolvable & otherwise. \end{cases}$$

2. If M is a closed term then there exists a term B_1 so that

$$B_1 \ulcorner \alpha \urcorner M = M^{\nu(\alpha)}.$$

Proof.

$$\begin{aligned} M^{\nu(\alpha)} &= (M^{\eta(\alpha)})^{\nu}, \\ B_1 \ulcorner \alpha \urcorner M &= Eqn(B_0 \ulcorner \alpha \urcorner M), \end{aligned}$$

where μ and Eqn are defined in 2.12.

□

Remark 2.18. In the preceding results we used the transformations. It is evident that we can use contexts instead, i.e. for any of the previous defined transformation μ there exists a context $C[\;\;]$ such that for any term M belonging to $\Lambda_f(\vec{x})$ $M^{\mu} = C[M]$.

The following definition and corollary are purely technical generalizations of the preceding result to finite sets of nodes.

Definition 2.19. *Let $S \subseteq Seq$ be finite and $M, N \in \Lambda_f^T(\vec{x})$.*

$$M \sim_S N \quad \text{if} \quad \forall \alpha \in S\, M \sim_\alpha N.$$

We shall denote $M|_S \downarrow$ if $\forall \alpha \in S\, M|_\alpha \downarrow$. An effective enumeration of finite sets of sequences is supposed. The number assigned to S will be denoted by $\natural S$ and $\ulcorner S \urcorner$ will be written instead of $\ulcorner \natural S \urcorner$. 2.16 can be generalized to finite sequences of nodes (in most cases finite trees).

Corollary 2.20. *For all finite $S \subset nodes(T)$ there exists a term B such that for any closed terms M and N such that $M, N \in \Lambda_f^T(\vec{x})$ and $M|_S \downarrow$, $N|_S \downarrow$ we have $B \ulcorner S \urcorner M = B \ulcorner S \urcorner N$ if an only if $M \sim_S N$.*

Proof. The assertion is implied by the following facts:

1. There exists a one–to–one general recursive function φ such that $\varphi(\natural\{\alpha_1, \ldots, \alpha_n\}) = \langle \natural\alpha_1, \ldots, \natural\alpha_n \rangle$. Let Φ denote the term which represents φ in λ–calculus.

2. There exists a term B' such that

$$B'\langle\ulcorner\alpha_1\urcorner,\ldots,\ulcorner\alpha_n\urcorner\rangle M=\langle\ulcorner\langle\alpha_1,B_1\ulcorner\alpha_1\urcorner M\rangle\urcorner, \ldots,\ulcorner\langle\alpha_n,B_1\ulcorner\alpha_n\urcorner M\rangle\rangle$$

where B_1 is as in 2.17

To build B' we need a representation of finite sequences of integers in the λ–calculus, together with the *concatenation operator* and the *test of empty sequence,* as in [4]. H is the concatenation operator if

$$H\langle x_1,\ldots,x_m\rangle\langle y_1,\ldots,y_n\rangle = \langle x_1,\ldots,x_m,y_1,\ldots,y_n\rangle$$

is valid. Let Z denotes a test for the empty sequence, i.e. $Z\langle\ \rangle = \mathsf{T}$ and $Z\langle x_1,\ldots,x_n\rangle = \mathsf{F}$ is valid. Let τ denote a sequence. Now B' can be be defined as follows:

$$B'(H\langle\ulcorner\alpha\urcorner\rangle\tau)M=Z\tau(\langle\ulcorner\langle\ulcorner\alpha\urcorner,B_0\ulcorner\alpha\urcorner M\rangle\rangle)(H\langle\ulcorner\langle\ulcorner\alpha\urcorner,B_0\ulcorner\alpha\urcorner M\rangle\rangle(B'\tau)M).$$

3. There exists a term Num such that

$$Num\langle\ulcorner n\urcorner_1,\ldots,\ulcorner n\urcorner_k\rangle = \ulcorner\langle\ulcorner n\urcorner_1,\ldots,\ulcorner n\urcorner_k\rangle\urcorner.$$

This assertion is a straightforward generalization of the results of [9].

Now B is defined as follows:

$$B\ulcorner S\urcorner M = Num(B'(\Phi\ulcorner S\urcorner)M) = \ulcorner(B'(\Phi\ulcorner S\urcorner)M)\urcorner.$$

□

3. Sufficient conditions for separability

In this section we shall construct a discriminator for some sets of terms. The intuitive idea is the following. The Böhm out technique described in the preceding paragraph enables an extraction of the (numeral of the) $\sim_\alpha$-class number from the term (asuming that the subterm at the node α is solvable). Let $M \in \mathcal{M}$ and let there exists a finite path, say p, i.e. a finite sequence of nodes, such that a) $M_\alpha\downarrow$ at any node α of p, b) for any member $M' \in \mathcal{M}$, if $M' \in \mathcal{M}$ differs from M, then there exists a node β on the path such that the $\sim_\beta$-class numbers of M and M' differ, and $M'_\gamma\downarrow$ at any node γ of p that precedes or is equal to β. Then a discriminator for M can be easily constructed by extracting $\sim_\beta$-class numbers of the terms M and M' and comparing them.

An appropriate formulation of the notion of distinctiveness which can be used for infinite sets seems now standard and can be found in Ronchi [11]. Our Definition 3.1 is equivalent to it but, by our opinion, written in a more convenient form for using the Böhm-out technique (as formulated in 2.16) in constructing discriminators. A method for constructing discriminators similar to ours is used in a different context by Böhm et al. [7].

A natural definition of the finite path having the length n is a function $p : \{0,\ldots,n\} \to Seq$. We shall modify this definition so that the information about the length is contained in the definition of the function. Let $*$ denotes anything not being a sequence. A finite path is a total function $p : \mathbb{N} \to Seq \cup \{*\}$, such that there exists $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $i \leq n$ implies $p(i) \in Seq$ while $i > n$ implies $p(i) = *$. Hence the length of p is the maximal number n such that $p(n) \in Seq$. The length of the path p will be denoted $lh(p)$. We shall say the node $\alpha \in Seq$ lies on the path p if $\alpha = p(i)$ for some $i \in \mathbb{N}$. We assume that paths are *without cycles,* i.e. p is injective. Let $\chi : Seq \cup \{*\} \to \mathbb{N}$ be partial recursive (in an effective enumeration). The function $\langle p,\chi\circ p\rangle:\mathbb{N}\to(Seq\cup\{*\})\times\mathbb{N}$ will be called *labeled path* and denoted p^l, i.e. $p^l = \langle p,\chi\circ p\rangle$. The number m is the *label* of α if $p(i) = \alpha$ and $\chi\circ p(i) = m$ for some $i \in \{0,\ldots,n\}$. Informally a labeled path is a path having labeled milestones. Erasing labels of the labeled path p^l results the *underlying path* of p^l. More formally the underlying path of p^l is $fst\circ p^l$. The underlying path of p^l will be denoted p. For an unlabeled path p the terms M and N are *$\sim_p$-equivalent* if they are $\sim_\alpha$-equivalent with respect to each node α lying on the path p. A path p is *defined* in a term M if $M_\alpha\downarrow$ for any α on the path, i.e. in the $BT(M)$ the node α is not labeled by $\perp$. A path p is defined in a set of terms $\mathcal{M}$ if it is defined in any member of $\mathcal{M}$. We shall say the λ–term P (resp. P^l) *represents the path* p (resp. the labeled path p^l) if for any $i \in \mathbb{N}$ less or equal the length of p $P\ulcorner i\urcorner = \ulcorner\alpha\urcorner$ (resp.

$P^l\ulcorner i\urcorner = \langle \ulcorner\alpha\urcorner, \ulcorner m\urcorner\rangle = \lambda x.x\ulcorner\alpha\urcorner\ulcorner m\urcorner)$, where α is the $i-$th node on the path and m is its label. $*$ will denote the term such that $P\ulcorner i\urcorner = *$ for any i greater than the length of the path. The double meaning of $*$ does not lead to confusion, since it is obvious from the context which of them is used. Let $\mathcal{N}$ be a set of numerals. We shall suppose $\mathcal{N} \cup \{*\}$ is a separable set, i.e. distinguishable in the lambda calculus. Hence $P\ulcorner i\urcorner = *$ iff $p(i) = *$ (note the two different meanings of $*$).

Suppose $M \in \mathcal{M}$ and p is a path of length n. Let us denote p_i the initial segment of p having the length i, i.e. the path $p_i : \{0, \dots, i\} \to Seq$, and $p_i(j) = *$ for any $j > i$. Let us denote $[M]_i/\mathcal{M}'$ the $\sim_{p_i}$-equivalence class of the term M with respect to p_i in the set $\mathcal{M}' \subseteq \mathcal{M}$.

Definition 3.1. *p is a distinctive path for M in $\mathcal{M}$ if there exists a finite sequence of sets $\mathcal{M}_0, \mathcal{M}_1, \dots \mathcal{M}_n \subseteq \mathcal{M}$ such that*

1. *p_i is defined in $\mathcal{M}_i$ for each i, $0 \leq i \leq n$,*
2. *$\mathcal{M}_0 = [M]_0/\mathcal{M}$,*
3. *$\mathcal{M}_{i+1} = [M]_{i+1}/\mathcal{M}_i$,*
4. *$\mathcal{M}_n = \{M\}$.*

Intuitively: Walking along a distinctive path the terms are thrown over at milestones where their equivalence class numbers become different of M's one and at the end of the path only M is retained. Note that a distinctive path is not empty.

We shall say "p is a distinctive path for M" if $\mathcal{M}$ is fixed. We shall say that $\mathcal{M} \subseteq \Lambda$ *has distinctive paths* if each of its members has a distinctive path. A set of terms $\mathcal{M}$ will be called *distinct* if each of its members has a distinctive path in $\mathcal{M}$. The underlying unlabeled path of the distinctive labeled path will be called the *underlying distinctive path.* The minimal (labeled) tree containing the (labeled) distinctive tree will be called the *(labeled) distinctive tree.*

Note that p^l is a labeled distinctive path for $M \in \{M_1, \dots\} \subseteq \Lambda(\vec{x})$ iff it is a labeled distinctive path for $\lambda\vec{x}.M$ in $\{\lambda\vec{x}.M, \dots\} \subseteq \Lambda^0$. We shall call $\lambda\vec{x}.M$ the *closure* of M with respect to $\Lambda(\vec{x})$ and shall denote $\bar{\mathcal{M}}$ the set of closures (with respect to all free variables of $\Lambda_f^T(\vec{x})$) of the members of $\mathcal{M}$, i.e. $\bar{\mathcal{M}} = \{\lambda\vec{x}.M | M \in \mathcal{M}\}$ where $\{\vec{x}\} = FV\{\mathcal{M}\}$.

As in the previous section T will denote any tree and we will suppose that the orders and the degrees of subterms M_α of a given term M for $\alpha \in$ nodes(T) are bound by a general recursive function.

Let $Eq \in \Lambda$ be an equality test for the numeral system i.e.

$$Eq\ulcorner m\urcorner\ulcorner n\urcorner = \begin{cases} \mathsf{T}, & \text{if } m = n, \\ \mathsf{F}, & \text{if } m \neq n. \end{cases}$$

Let B_1 be as in 2.17 and suppose $M, N \in \Lambda_f^T(\vec{x})$, where T is a nonempty tree. Hence $Eq(B_1\ulcorner\alpha\urcorner(\lambda\vec{x}.M))(B_1\ulcorner\alpha\urcorner(\lambda\vec{x}.N)) = \mathsf{T}$ iff $M \sim_\alpha N$, asuming $M_{\alpha_i}\downarrow, N_{\alpha_i}\downarrow$ and $\alpha \in$ nodes$T$.

We shall generalize the preceding test of the equivalence at nodes to the *test of equivalence* on paths. Denote $\tilde{P}$ a term representing a path and L a term representing an algorithm that calculates the length of the path. An algorithm calculating the length of the path can be constructed as follows.

Calculate $\tilde{P}\ulcorner 0\urcorner, \dots, \tilde{P}\ulcorner i\urcorner, \dots$, while $\tilde{P}\ulcorner i\urcorner \neq *$ and return $\ulcorner n\urcorner$ if $\tilde{P}\ulcorner n+1\urcorner = *$.

A construction of the term representing the preceding algorithm is in Appendix. Hence L is a term such that $L\tilde{P}$ represents the preceding algorithm in $\lambda-$calculus. Let $\tilde{\Delta}$ be a solution of the equation

$$\begin{aligned} \tilde{\Delta} = \lambda uxyz.&\textit{If } Eq(B_1(xu)y)(B_1(xu)z) \\ &\textit{then } (\textit{If } Eq\,u\,(Lx) \\ &\quad \textit{then } \mathsf{T} \\ &\quad \textit{else } (\tilde{\Delta}(R^+u)xyz)) \\ &\textit{else } \mathsf{F}. \end{aligned} \tag{3.1}$$

Hence

$$\begin{aligned} &\tilde{\Delta}u\tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) \\ &= \textit{If } Eq(B_1(\tilde{P}u)(\lambda\vec{x}.M))(B_1(\tilde{P}u)\,(\lambda\vec{x}.N)) \\ &\quad \textit{then } (\textit{If } Eq\,u\,(L\tilde{P}) \\ &\quad\quad \textit{then } \mathsf{T} \\ &\quad\quad \textit{else } (\tilde{\Delta}(R^+u)\tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N))) \\ &\quad \textit{else } \mathsf{F}. \end{aligned} \tag{3.2}$$

We shall prove that $\tilde{\Delta}\ulcorner 0\urcorner$ is a test of equivalence on paths.

Lemma 3.2. *1. If $M \sim_\alpha N$ and $M_\alpha \downarrow$ for any node α of the path $\tilde{p}$ then*

$$\tilde{\Delta}\ulcorner 0\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) = \mathsf{T}.$$

2. If exists a node $\alpha = \tilde{p}(i)$ on the path $\tilde{p}$ such that: a) $M \not\sim_\alpha N$, b) $M_\beta \downarrow$ and $N_\beta \downarrow$ at any node $\beta = \tilde{p(j)}$, $0 \leq j \leq i$ then

$$\tilde{\Delta}\ulcorner 0\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) = \mathsf{F}.$$

Proof. Suppose $lh(\tilde{p}) = n$, i.e. $L\tilde{P} = \ulcorner n\urcorner$. The following equation is a consequence of equation 3.2.

$$\begin{aligned}&\tilde{\Delta}\ulcorner i\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N)\\ &\quad = If\ Eq(B_1(\tilde{P}\ulcorner i\urcorner)(\lambda\vec{x}.M))(B_1(\tilde{P}\ulcorner i\urcorner)(\lambda\vec{x}.N))\\ &\qquad then\ (If\ Eq\ulcorner i\urcorner\ulcorner n\urcorner\\ &\qquad\quad then\ \mathsf{T}\\ &\qquad\quad else\ (\tilde{\Delta}(\ulcorner i+1\urcorner)\tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N)))\\ &\qquad else\ \mathsf{F}.\end{aligned}$$

Suppose a) $\alpha = \tilde{P}(i)$ where $0 \leq i \leq n$, b) $M_\beta \downarrow$, $N_\beta \downarrow$ and $M \sim_\beta N$, for any β that precedes α on the path.
Case 1: $i < n$ and $M \sim_\alpha N$.
Hence

$$\tilde{\Delta}\ulcorner i\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) = \tilde{\Delta}\ulcorner i{+}1\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N),$$

by induction on i.
Case 2: $i = n$ and $M \sim_\alpha N$.

$$\tilde{\Delta}\ulcorner n\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) = \mathsf{T},$$

by Case 1.
Case 3: $M \not\sim_\alpha N$.

$$\tilde{\Delta}\ulcorner i\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) = \mathsf{F}.$$

Hence $\tilde{\Delta}\ulcorner 0\urcorner \tilde{P}(\lambda\vec{x}.M)(\lambda\vec{x}.N) = \mathsf{T}$ (resp. F) if $M \sim_{\tilde{p}} N$ (resp. $M \not\sim_{\tilde{p}} N$).

□

Let us suppose $M \in \mathcal{M} \subseteq \Lambda$ has a distinctive path, say p. Obviously for any $N \in \mathcal{M}$, if $N \neq M$ then there exists an initial segment q of p such that $M \not\sim_q N$. Let us suppose a distinctive path for $M \in \mathcal{M}$ is fixed. For any term $N \in \mathcal{M} \subseteq \Lambda$, $\tilde{p}_N$ will denote:

- the distinctive path, if $N = M$,
- an initial segment of the distinctive path $\tilde{p}_M$ such that $M \not\sim_{\tilde{p}_N} N$, if $N \neq M$.

We can assume that $\tilde{p}_N$ is the shortest such initial segment, i.e. $M \sim_q N$ where q is an initial segment of $\tilde{p}_M$ whose length is less that the length of $\tilde{p}_N$ is. Let us denote T_N the tree that spans the nodes of $\tilde{p}_N$.

The following example shows an idea for constructing a discriminator.

Example 3.3. *Let $A \equiv \lambda xy.xx$, $B \equiv \lambda xy.y\Omega$, $C_n \equiv \lambda xy.y(K^n I)$, for $n = 0, 1, \ldots$ and $D \equiv \lambda xy.xy$. A distinctive path p_A for A in $\mathcal{M} = \{A, B, D\} \bigcup \{C_n | n{=}0, 1 \ldots\}$ is equal to $\{\langle\rangle, \langle 1\rangle\}$. Obviously $\tilde{p}_B{=}\tilde{p}_{C_n}{=}\{\langle\rangle\}$ and $\tilde{p}_D{=}\{\langle\rangle, \langle 1\rangle\}$. A discriminator for A in $\mathcal{M}$ is a λ–representation of the following algorithm:*

$$\begin{aligned}&If\ N \not\sim_{\langle\rangle} A\quad then\ \mathsf{F}\\ &\qquad else\ if\ N \not\sim_{\langle 1\rangle} A\quad then\ \mathsf{F}\\ &\qquad\qquad else\ \mathsf{T},\end{aligned}$$

where N is any term of $\mathcal{M}$. Note that using the Böhm-out technique a representation of the preceding algorithm in the λ–calculus that will work on $\mathcal{M}$, can be built. Obviously $B, C_n \in (\Lambda_f^{\{\langle\rangle\}})^0$ and $A, D \in (\Lambda_f^{\{\langle\rangle,\langle 1\rangle\}})^0$. Note that building a discriminator using the Böhm-out technique, the condition $N \in (\Lambda_f^{T_N})^0$, where f is an appropriate general recursive function ($f(n) = \langle 2, 1\rangle$ in the present example), requires for all $N \in \mathcal{M}$.

Proposition 3.4. *Let $\mathcal{M} \subseteq \Lambda(\vec{x})$ and f be a general recursive function. Suppose*

1. *$M \in \mathcal{M}$ has a distinctive path,*
2. *for any $N \in \mathcal{M}$ $N \in \Lambda_f^{T_N}(\vec{x})$.*

Then $\lambda\vec{x}.M$ has a discriminator in $\bar{\mathcal{M}}$.

Proof. Obviously each T_N is nonvoid. Let $P_M \in \Lambda$ be a term representing the distinctive path of M. Hence

$$D_M = \tilde{\Delta}\ulcorner 0\urcorner P_M M$$

is a discriminator for $\lambda\vec{x}.M \in \bar{\mathcal{M}}$ by 3.2.

□

Remark 3.5. Evidently D_M is a discriminator for M in $M \in \mathcal{M} \subseteq \Lambda(\vec{x})$ implies $\lambda y.D_M(y\vec{x})$ is a discriminator for $\lambda\vec{x}.M$ in the set $\bar{\mathcal{M}}$.

An inspection of the construction in the preceding proof shows $D_M N = D_M \tilde{N}$, for any term $\tilde{N}$ meeting the demand $N \sim_{\tilde{p}_N} \tilde{N}$. In other words $D_M N$ depends only on $\sim_\alpha$–class numbers for α's that lies on the (unlabeled) distinctive path. If anything out of it is changed in the (the Böhm trees of) terms, the discriminator does not change. Hence there exist separable sets of terms which are not sequences of terms, since any sequence is r.e. while discriminable or separable may be not r.e.

Example 3.6. *$\lambda z.zMN$ is an ordered pair in λ–calculus having the first projection $\Pi_1 = \lambda t.t\mathsf{T}$.*

Let $\mathcal{M}$ be the set $\{\lambda z.zc_n M'_n | n \in \mathbb{N}\}$, where $c_n = \lambda xf.f^n x$ are Church numerals and M'_n are any terms. Let us denote $M_n = \lambda z.zc_n M'_n$. Two examples of subterms seating on the nodes of the length 1 are as follows: $M_0|_{\langle 1\rangle} = \lambda x.x$ and $M_0|_{\langle 2\rangle} = M'_0$. The distinctive path for the term M_n is p_n defined as follows: $p_n(0) = \langle 1,1\rangle, \ldots$, $p_n(i) = \langle 1,\ldots,1\rangle$ ($i+1$ 1's) for $0<i\leq n$, $p_n(i) = *$, for $i>n$. The tree T_{M_n} is the tree having the set of nodes $\{\langle\rangle, \langle 1\rangle, \langle 1,1\rangle, \ldots\}\langle 1,\ldots,1\rangle$ ($n+1$ 1's). Since terms M'_n, $n = 0,1,\ldots$ are out of T they can be chosen freely. The discriminator for M_n = is $\lambda x.Eq_c(\Pi_1 x)c_n$, where Eq_c is an equality test for Church numerals. It is obvious that $\mathcal{M}$ is separable.

The following example shows that the condition $M \in \Lambda_f^T(\vec{x})$ is not necessary for having a discriminator.

Example 3.7. *$\lambda x.x\mathsf{T}$ is a discriminator for $\lambda x.x$ in $\mathcal{M} = \{\lambda x.x, \lambda x.x\mathsf{FF}, \lambda x.x\mathsf{FFFF}, \ldots\}$. The distinctive path of $\lambda x.x$ is $\langle\,\rangle$, i.e. $T = T_N = \langle\,\rangle$ for any $N \in \mathcal{M}$. It is obvious $\mathcal{M} \not\subset \Lambda_f^T(\vec{x})$ for any f since the degrees of its members grow into infinity.*

The discriminator as built in 3.2 has a particular shape. In general the result of the application of a discriminator to a term can depend on subterms lying out of the distinctive path, as the previous example shows.

Suppose each member of $\mathcal{M} \subset \Lambda$ has a distinctive path. Denote f_M a function corresponding to the distinctive path for M so that $N \in \Lambda_{f_M}^{T_N}(\vec{x})$. Suppose there exists $f = \sup_{M\in\mathcal{M}} f_M$. Hence the construction of the Böhm-out term and the construction of $\tilde{\Delta}$ can be made uniformly with respect to f. Denote T_{MN} the tree that spans the nodes of an initial segment p_{MN} of the distinctive path p_M for $M \in \mathcal{M} \subset \Lambda$ such that $N \not\sim_{p_{MN}} M$. Let $T = \bigcup_{M,N\in\mathcal{M}} T_{MN}$.

Corollary 3.8. *If any member of $\mathcal{M} \subset \Lambda_f^T(\vec{x})$ has a distinctive path in $\mathcal{M}$ then $\bar{\mathcal{M}} = \{\lambda\vec{x}.N | N \in \mathcal{M}\}$ has discriminators.*

Proof. Obviously $T_2 \subseteq T_1$ imply $\Lambda_f^{T_1}(\vec{x}) \subseteq \Lambda_f^{T_2}$ Hence the assumptions for by 3.4 are accomplished for any $M \in \mathcal{M} \subset \Lambda_f^T(\vec{x})$

□

4. Necessary conditions for separability

In this section we shall construct distinctive paths for discriminable sets. The first step is a slight generalization of Wadsworth's $\lambda\bot$ calculus. The idea is to mark the $\bot$'s by labels of the nodes on which this $\bot$ seat in the $BT(M)$, and then trace which $\bot$ in the term M causes $\bot$ in the term FM. The construction will be based on Wadsworth's ω–normal forms in $\lambda\bot$–calculus ([1] 14.3.1. and 14.3.6.) which is used for constructing an approximation of a term in the Scott's topology. We shall add labels of the form $\langle\alpha,1\rangle$ and $\langle\alpha,2\rangle$ to the tokens $\bot$. The idea is as follows: for a given finite tree T α marks a node while the second index marks which of the two possible alternative reasons caused a (sub)term reduced to $\bot$: if the node α lies out of T or the (sub)term is not in *hnf*. These marks will be used for constructing the distinctive path for a term $M \in \mathcal{M}$ where $\mathcal{M}$ is discriminable. The idea for constructing the distinctive path is as follows: suppose in $BT(M)$ some labels are replaced by (marked) $\bot$'s, hence the resulting tree is $BT(M')$ for a term $M' \sqsubseteq M$. If the application of the discriminator results T or F then it is the discriminator for M' too. If not we can replace the $\bot$ in $BT(M')$ which causes the $\bot$ in FM by the label of $BT(M)$ and put $\bot$'s for its sons. The information of the label can be extracted by 2.17. Since FM has an ω–normal form the algorithm stops after a finite number of steps. Beginning at the root node the algorithm gives the minimal term for which D is a discriminator. The soundness of the algorithm is based on the continuity and sequentiality theorem of the λ-calculus.

Let $L = \{\langle\alpha, i\rangle | \alpha \in Seq,\ i \in \{1,2\}\}$. We shall denote:

1. $\bot_l$, where $l \in L$, a $\bot$ labeled by l,
2. $\Lambda\bot_L$ the set of terms built of variables, $\bot_l$'s for $l \in L$, abstraction and application.

The members of $\Lambda\bot_L$ will be called $\lambda\bot_L$–terms. A definition of the $\lambda\bot_L$–calculus is as follows.

Definition 4.1. *1. $\lambda\bot_L$–terms are recursively defined as follows:*

$$M ::= x \quad | \quad \bot_l \quad | \quad M_1M_2 \quad | \quad \lambda x.M_1,$$

2. $\beta\bot_L$–reduction:

$$\begin{array}{ll} \beta : & (\lambda x.M)N \to_\beta M[x := N], \\ \bot_L : & \lambda x.\bot_l \to_\bot \bot_l \text{ and } \bot_l M \to_\bot \bot_l, \\ \beta\bot_L : & \beta \cup \bot_L. \end{array}$$

3. Substitution:

$$\begin{array}{rcl} \bot_l[x := M] &=& \bot_l \\ x[x := M] &=& M, \\ y[x := M] &=& y \qquad \text{if } \ y \neq x, \\ MN[x := R] &=& (M[x := R])(N[x := R]) \\ (\lambda y.M)[x := N] &=& \lambda y.(M[x := N]) \\ && \text{if } N \text{ is free for } x \text{ in } M. \end{array}$$

The proof that $\beta\bot_L$–reduction is weakly Church–Rosser and that each term has a unique $\beta\bot_L$–normal form is the same as for $\lambda\beta$–calculus. A characteristic example is

$$\begin{array}{l} (\lambda x.\bot_l)M \to_\beta \bot_l[x := M] \equiv \bot_l, \\ (\lambda x.\bot_l)M \to_{\bot_L} \bot_l M \to_\bot \bot_l. \end{array}$$

$\bot_{\langle\alpha,i\rangle}$ will be written $\bot_{\alpha,i}$.

Λ can be embedded into $\Lambda\bot_L$ in an obvious way. Note the $\beta\bot_L$–normal form of any term $M \in \Lambda \subset \Lambda\bot_L$ coincides with its β–normal form (normal form in the pure λ–calculus). Nevertheless we shall write "β–normal form instead" of "normal form" whenever want to emphasize it is $\bot_l$–free.

The following definition is a variant of [1] 14.3.6 for labeled $\bot$'s.

Definition 4.2. *Let T be a finite tree and $\alpha \in Seq$. The ω_T^α–normal form of the term $M \in \Lambda\bot_L$ is recursively defined as follows.*

$$\begin{array}{l} \omega_T^\alpha(\bot_l) = \bot_{\alpha,1}, \\ \omega_T^\alpha(\lambda\vec{x}.y) = \lambda\vec{x}.y \\ \omega_T^\alpha(\lambda\vec{x}.yM_1\ldots M_n) = \lambda\vec{x}.yN_1\ldots N_n \end{array}$$

where

$$N_i = \begin{cases} \omega_T^{\alpha*\langle i\rangle}(M_i), & \text{if } \alpha * \langle i\rangle \in nodes\ (T), \\ \bot_{\alpha*\langle i\rangle,1} & \text{otherwise.} \end{cases}$$

$$\omega_T^\alpha(\lambda\vec{x}.(\lambda y.P)Q\vec{M}) = \bot_{\alpha,2}.$$

Informally: if $\bot_l$ is contained in $\omega_T^{\langle\rangle}(M)$, where $l = \langle\alpha, i\rangle$, then α marks the position of $\bot_l$ in the Böhm tree of $\omega_T^{\langle\rangle}(M)$, while the meaning of i is as follows: if $M|_\alpha = \bot_l$ or $M|_\alpha$ lies out of T then $i = 1$, if the subterm at the node α of the 'partially developed' Böhm tree of M is not in head normal form then, $i = 2$.

Example 4.3. *Let $M = \lambda x.x(xI)((\lambda y.y)zz)\bot_{\langle 1\rangle,2}x$ and $T = \{\langle\,\rangle, \langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 1,1\rangle\}$.*

$$\begin{array}{l} \omega_T^{\langle\rangle}(\lambda x.x(xI)((\lambda y.y)zz)\bot_{\langle 1\rangle,2}x) \\ \quad = \lambda x.x\omega_T^{\langle 1\rangle}(xI)\omega_T^{\langle 2\rangle}((\lambda y.y)zz)\omega_T^{\langle 3\rangle} \\ \qquad\qquad (\bot_{\langle 1\rangle,2})\bot_{\langle 4\rangle,1} \\ \quad = \lambda x.x(x\omega_T^{\langle 1,1\rangle}(I))\bot_{\langle 2\rangle,2}\bot_{\langle 3\rangle,1}\bot_{\langle 4\rangle,1} \\ \quad = \lambda x.x(xI)\bot_{\langle 2\rangle,2}\bot_{\langle 3\rangle,1}\bot_{\langle 4\rangle,1}. \end{array}$$

Instead of $\omega_T^{\langle\,\rangle}$ we shall write ω_T. It is evident that ω_T^α is a $\beta\bot_L$–normal form. Remember that $BT^n(M)$ denotes the tree resulting from $BT(M)$ when it is pruned at depth n and $M^{[n]}$ denotes a term whose Böhm tree is $BT^n(M)$. Let $\sqsubseteq$ denote the relation of approximation between terms, i.e $M \sqsubseteq N$ iff $BT(M) \subseteq BT(N)$. Note that labels of $\bot$ do not play any role in the definition of $\sqsubseteq$, for example $\bot_l \sqsubseteq \bot_m$ for any l and m. The ω_T-normal forms, belonging to the same equivalence class of the equivalence relation $\cong$ induced by the preorder $\sqsubseteq$ ($M \cong N$ iff $M \sqsubseteq N$ and $M \sqsupseteq N$) are exactly the ω_T-normal forms which differ only for the labels of $\bot$'s. Hence the partial order relation $\sqsubseteq$ modulo $\cong$ is the well known partial order on $\lambda\bot$-terms and so the results of this theory will be applied freely. We shall use the same symbol $\sqsubseteq$ for both relations.

It is evident that if the depth of the tree T is less or equal n then $\omega_T(M) \sqsubseteq M^{[n]}$. Let Γ denote a λ–term having a normal form whose Böhm tree has at most one son at each node. The last condition is not essential for our construction but makes it simpler. Some terms of this kind that will be used are *Church numerals* $c_n \equiv \lambda f x.f^n x$, and combinators denoting true and false, i.e. T and F.

We shall say that $FM|_\alpha$ is *caused* by some $M|_\beta$, if $FM|_\alpha = \bot_l$ and for any $M' \sqsupseteq M$ $FM'|_\alpha = \bot_l$ assuming $M'|_\beta = \bot_{l'}$ (in fact $l = l'$, see later) and $FM'|_\alpha \neq \bot_l$ assuming $M'|_\beta = z$ for a fresh variable z. The term M is α*–constant* if for any $M' \sqsupseteq M$ $FM'|_\alpha = \bot_l$ (see [1] 14.4.2). We shall say $\bot_{l'}$ at the node β of the term $\omega_T(M)$ is *created* by the application of ω_T on M if $\omega_T(M)|_\beta = \bot_{l'}$, while $M|_\beta \neq \bot_l$ for all $l \in L$.

$C[\ \ ]$ will denote a context. By "context" we ever mean a multiple context, i.e. a term having a finite number of holes. We shall denote by $C[\vec{M}]$ or $C[M_1, \ldots, M_n]$ the term that results by substituting a string of terms $M_1, \ldots, M_n$ for all holes of $C[\ \ ]$. Whenever $M = M_1, \ldots, M_n$ we shall write $C[M]$ instead of $C[\vec{M}]$ or $C[M, \ldots, M]$. This notation is convenient for a unified treating of terms and contexts. If M is closed, then for any context $C[\ \ ]$ there is a term Q so that $QM = C[M]$. The statement "$C[M]|_\alpha = \bot_l$ is caused by some $M|_\beta$", will mean $C[M]|_\alpha = \bot_l$ is caused by some $M|_\beta$ in some copy of $M|_\beta$ etc. Note the double meaning of $[\ \ ]$: $[\ \ ]$ in $C[\ \ ] = \lambda x.x[\ \ ][\ \ ]$ is a string of holes while $[\ \ ]$ in $\lambda x.[\ \ ]C_1 \ldots C_n$ is a single hole.

Lemma 4.4. *Let $C[\ \ ]$ be any context and M be any term such that $C[M] = \Gamma$ where Γ is a normal form of the λ–calculus. Then*

1. $C[\omega_T(M)] \sqsubseteq \Gamma$,
2. $T_1 \subseteq T_2 \quad \Rightarrow \quad C[\omega_{T_1}(M)] \sqsubseteq C[\omega_{T_2}(M)]$,
3. $M \to N \quad \Rightarrow \omega_T(M) \sqsubseteq \omega_T(N)$,
4. $\exists T$, T finite $\wedge\ C[\omega_T(M)] = \Gamma$,
5. $\forall T$, T finite $\Rightarrow$ $C[\omega_T(M)]|_\alpha = \bot_l$ *is caused by some* $\omega_T(M)|_\beta$, *which is created by the application of* ω_T *on* M.

Proof.

1. Since $\omega_T(M) \sqsubseteq M$ the assertion is implied by [1] 14.3.20 (iii).
2. Analogously as (1) since $T_1 \subseteq T_2 \quad \Rightarrow$ $\omega_{T_1}(M) \sqsubseteq \omega_{T_2}(M)$.
3. This is [1] 14.3.7.
4. By [1] 14.3.19. and [1] 14.3.20. there exists an increasing sequence of finite trees $T_1, T_2, \ldots$ such that

$$C[M] = \Gamma = \sup_n\{F(\omega_{T_n}(M)) | n \in \mathbb{N}\}.$$

   Since Γ is a normal form there exists an m such that $C[\omega_{T_m}(M)] = \Gamma$.

5. $C[\omega_T(M)]|_\alpha$ is not α–constant since $C[M]$ has a β–normal form, i.e. a $\bot_l$–free normal form. Hence $C[\omega_T(M)]|_\alpha$ is caused by some $\omega_T(M)|_\beta$ (see [1] 14.4.8). There are two possibilities: a) $\omega_T(M)|_\beta = \bot_l$ since $M|_\beta = \bot_l'$ and b) $\omega_T(M)|_\beta$ is created by the application of ω_T on M. Since $C[M]$ has a β–normal form only the second of the preceding two possibilities can happen.

□

The assertion 4.4 (5) can be sharpened. It is obvious that the subscript of $\bot_l$ doesn't change when $\beta\bot_l$–reductions are executed. Hence in the label of $C[\omega_T(M)]|_\alpha$ gives us the information which $\omega_T(M)|_\beta$ causes $C[\omega_T(M)]|_\alpha$. The following proposition corresponds to [1] 14.4.4. for labeled $\bot$ ' s.

Proposition 4.5. *Let $C[\omega_T(M)]|_\alpha$ be caused by some $\omega_T(M)|_\beta$. Then $\omega_T(M)|_\beta = \bot_l$ implies $F(\omega_T(M)|_\alpha) = \bot_l$.*

Now we define a syntactical algorithm for the distinctive path construction. Let Γ be $\Gamma =$ T or $\Gamma =$ F or $\Gamma = \lambda f x.f^n x$, hence 4.4 applies to M and $C[\omega_{T_n}(M)]$ contains at most one $\bot_l$.

Definition 4.6. *Let $C[\ \ ]$ be any context and M be any term such that $C[M] = \Gamma$. We define recursively a finite sequence of trees $T_0, T_1, \ldots$*

and a finite sequence of terms $M_0, M_1, \ldots$ as follows.

$$T_0 = \langle\rangle$$
$$M_0 = \omega_{T_0}(M),\ \text{i.e. } M_0{=}\lambda\vec{y}.z\bot_{\langle 1\rangle,1}\cdots\bot_{\langle s\rangle,1}$$
$$\text{or}\quad M_0 = \bot_{\langle\rangle,2}.$$

If $C[M_k] = \Gamma$ then the process is finished; else $C[M_k] \sqsubseteq \Gamma$ contains exactly one subterm of the shape $\bot_{\alpha,i}$.
If $i = 1$ then

$$T_{k+1} = T_k \cup \{\alpha\},$$
$$M_{k+1} = \omega_{T_{k+1}}(M).$$

If $i = 2$ then $M|_\alpha$ is not in hnf. Make a step of a head β–reduction on α: $M \to_\beta M'$ and put

$$T_{k+1} = T_k,$$
$$M_{k+1} = \omega_{T_{k+1}}(M').$$

Example 4.7. *Let $M = \lambda x.x\mathsf{F}\mathsf{T}(I\mathsf{T})$, and $C[\ \ ] = [\ \ ]I$, hence $FM = C[M]$ for $F = \lambda y.yI$.*

$T_0{=}\{\langle\rangle\}$, $\quad M_0{=}\lambda x.x\bot_{\langle 1\rangle,1}\bot_{\langle 2\rangle,1}\bot_{\langle 3\rangle,1}$, $FM_0{=}\bot_{\langle 1\rangle,1}$,

$T_1{=}\{\langle\rangle,\langle 1\rangle\}$, $\quad M_1{=}\lambda x.x\mathsf{F}\bot_{\langle 2\rangle,1}\bot_{\langle 3\rangle,1}$, $FM_1{=}\bot_{\langle 3\rangle,1}$,

$T_2{=}\{\langle\rangle,\langle 1\rangle,\langle 3\rangle\}$, $M_2{=}\lambda x.x\mathsf{F}\bot_{\langle 2\rangle,1},\bot_{\langle 3\rangle,2}$, $FM_2{=}\bot_{\langle 3\rangle,2}$,

$T_3{=}\{\langle\rangle,\langle 1\rangle,\langle 3\rangle\}$, $M_3{=}\lambda x.x\mathsf{F}\bot_{\langle 2\rangle,1}\mathsf{T}$, $FM_3{=}\mathsf{T}$.

We shall denote:

1. $\tilde{T}_M$ the last member in the sequences $T_0, T_1, \ldots$ in 4.6,
2. $\tilde{M}$ the term obtained by replacing $\Omega = (\lambda x.xx)(\lambda x.xx)$ to all $\bot$'s in the last member of the sequences $M_0, M_1, \ldots$
3. $\tilde{p}_M$ (resp. $\tilde{p}^l_M$) the path (resp. the path labeled by $\sim_\alpha$ –equivalence class numbers of M) as constructed by the algorithm in the definition 4.6, i.e. $\tilde{p}_M(n) = \alpha_n$, where α_n is the node constructed in the n'th step.

Let Γ, Γ_1 and Γ_2 denote Church numerals or T or F.

Lemma 4.8. *1. The algorithm defined in 4.6 stops in a finite number of steps.*

2. $\tilde{M}$ is the $\sqsubseteq$–minimal term such that $C[\tilde{M}] = \Gamma$ is accomplished.

3. $\forall\alpha \in nodes(\tilde{T}_M)$ we have $M_\alpha \downarrow$.

Proof.

1. 4.4 (2) and (3) imply that the sequence $C[M_k]$, $k = 0, 1, \ldots$ is $\sqsubseteq$-growing. 4.4 (4) implies that only a finite number of steps of the first type where a new node is added to the tree T_k is possible.

   If a term $M_k|_\alpha$ has a principal *hnf* it reduces to it by a finite number of head reductions. Hence the number of steps of the second type in the preceding definition is finite in this case. If a term $M_k|_\alpha$ does not have a principal *hnf* then it is unsolvable, hence $BT(M)|_\alpha = \bot_l$. Hence a $\bot_{l'}$ in $C[M]$ is caused by $M|_\alpha$ which is impossible since $C[M]$ has a β–normal form.
2. This is evident from the construction of algorithm defined in 4.6.
3. This is in the proof of (1).

□

An η–expansion of the term M is $\lambda\vec{y}.M\vec{y}$, where all variables in $\vec{y}$ are fresh. Denote $\vec{M}^\eta$ a string of terms as follows: M_i^η is obtained from M_i by replacing some subterms $(M_i)_{\alpha_i}$ by their η–expansions, where $\alpha_i \in Seq$, $1 \le i \le m$.

Fact 4.9. *Let $C[\ \ ]$ be a context so that $C[\vec{M}]$ reduces to T (resp. F) for some string of terms $\vec{M}$. Then for any finite sequence of sequences $\vec{\alpha}$ and any η–expansions of some subterms of $\vec{M}$, $C[\vec{M}^\eta]$ reduces to an η–expansion of T (resp. F).*

This fact is easy to prove. For example, let $C[\ \ ] = \lambda x.[\ \ ]x$ and $M = \lambda xy.y$. Hence $C[\lambda uvw.Muvw] = \lambda x.[\lambda uvw.vw]x = \lambda xvw.vw$ A formal proof is as follows.

Proof of 4.9. Let $=_\eta$ denote the equality relation in $\beta\eta$–calculus, i.e. the reflexive, symetric and transitive closure of the $\beta\eta$–reduction.
Case A: $\vec{\alpha} = \langle\ \rangle, \ldots, \langle\ \rangle$. Hence M^η is an η–expansion of M. The assertion is accomplished since $\vec{M}_1 =_\eta \vec{M}_2$ implies $C[\vec{M}_1] =_\eta C[\vec{M}_2]$.

Case B: Suppose that $\alpha_1, \ldots, \alpha_n$ is any finite sequence of sequences. Let us substitute a hole for each M_{α_i} in M_i. Hence there are, say, m holes in n terms M_i. Denote the constructed contexts $M_i[\ \ ]$. Denote $(\vec{M_i})_{\vec{\alpha_i}}$ the sequence of terms such that $M_i = M_i[(\vec{M_i})_{\vec{\alpha_i}}]$. Define $C_1[\ \ ] = C[M_1[\ \ ], \ldots, M_n[\ \ ]]$. Hence $C[\vec{M}^\eta]=C[M_1[(\vec{M_1})^\eta_{\vec{\alpha_1}}, \ldots (M_n)^\eta_{\vec{\alpha_n}}]]=C_1[(M_1)^\eta_{\alpha_1}, \ldots (M_m)^\eta_{\alpha_m}]$ reduces to an $\eta-$expansion of T (resp. F) by A. □

It is easy to see that the set $\{\lambda\vec{y_1}.\mathsf{T}\vec{y_1}, \lambda\vec{y_2}.\mathsf{F}\vec{y_2}\}$ is separable. Hence the preceding Fact implies that $\{M^\eta, N^\eta\}$ is separable presuming that $\{M, N\}$ is so. A consequence of the preceding fact is that a substitution of M_α by some of its $\eta-$expansion does not affect the discriminability.

Lemma 4.10. *Let*

1. *$C[\ \ ]$ be a context and $M, N \in \Lambda$ be any terms so that $C[M] = \mathsf{T}$ and $C[N] = \mathsf{F}$,*
2. *$\tilde{p}_M$ be the path constructed by 4.6 for the term M.*

Then there exists an initial segment $\tilde{p}'$ of the path $\tilde{p}_M$ having the length, say, n' such that

1. *$N|_{\tilde{p}'} \downarrow$,*
2. *$N \sim_\alpha M$ for any $\alpha = \tilde{p}'(i)$, where $0 \leq i < n'$,*
3. *$M \not\sim_\alpha N$ for $\alpha = \tilde{p}'(n')$.*

Proof. By induction on the length of the path $\tilde{p}_M$. M is solvable otherwise $C[M] = C[N] = C[\bot]$ contradicting the assumption $C[M] = \mathsf{T}$ and $C[N]=\mathsf{F}$. Hence it has a *hnf* say $\lambda\vec{y}.zM_1 \ldots M_s$. Hence the first term constructed by 4.6 is as follows:

$$\omega_T(M) = \lambda y_1 \ldots y_r.z\bot_{\langle 1\rangle,1} \cdots \bot_{\langle s\rangle,1}$$

for $T = \{\langle\,\rangle\}$. Analogously N has a *hnf* having the order r' and the degree s'. We can suppose $r' \leq r$, otherwise we interchange M and N. Denote

$$N' = \lambda y_1 \ldots y_r.z'N_1 \ldots N_{s'}y_{r'+1} \cdots y_r$$

an $\eta-$expansion of N having the same order as the *hnf* of M. Hence the set $\{M, N'\}$ is separable.

Let $lh(\tilde{p}_M) = 0$. Then

$$C[\omega_T(M)]=C[\lambda y_1 \ldots y_r.z\bot_{\langle 1\rangle,1} \cdots \bot_{\langle s\rangle,1}]=\mathsf{T}.$$

Suppose $M \sim N$. Since M and N' have the same order, they have the same degree and the same head variable, i.e. $\omega_T(M) = \omega_T(N') = \lambda y_1 \ldots y_r.z\bot_{\langle 1\rangle,1} \cdots \bot_{\langle s\rangle,1}$. Hence $C[\omega_T(M)] = C[\omega_T(N')] = \mathsf{T}$, contradicting the presumption that $\{M, N'\}$ is separable.

Let $lh(\tilde{p}_M) = n > 0$. As in the case $n = 0$, we have that $M \sim N$ implies $C[\omega_T(M)] = C[\omega_T(N')] = \bot_{\langle j\rangle,1}$ for some $1 \leq j \leq r$, since $C[\omega_T(M)] \sqsubset \mathsf{T}$ and $\bot$ is the only term accomplishing the relation $\bot \sqsubset \mathsf{T}$. Let

$$C'[\ \ ] = C[\lambda y_1 \ldots y_r.z\bot_{\langle 1\rangle,1} \cdots \bot_{\langle i-1\rangle,1}[\ \ ]\bot_{\langle i+1\rangle,1} \cdots \bot_{\langle s\rangle,1}].$$

Then $C'[M_i] = C[M]$ and $C'[N_i] = C[N']$ ($N_i = y_{i-s'+r'}$ for $i > s'$), i.e. the set $\{M_i, N_i\}$ is separable. Since the length of the path $\tilde{p}_{M_i}$ is $n-1$, the assertion of the lemma is accomplished for M_i, N_i by the induction hypothesis. Hence the assertion of the lemma is accomplished for M, N since $\tilde{p}'$ is constructed from the corresponding path for M_i, N_i by adding a node at the beginning.

□

Corollary 4.11. *If M has a discriminator in $\mathcal{M} \subset \Lambda^0$ then M has a distinctive path.*

Proof. Suppose D is a discriminator for M in $\mathcal{M}$. Define a context $C[\ \ ]$ such that $C[N] = DN$ for any closed term N. Evidently the previous lemma implies that the path constructed by the algorithm 4.6 is the distinctive path for M in $\mathcal{M}$. □

5. Application to Complete Discriminability and Separability

Let $P_M \in \Lambda$ be a term representing the (unlabeled) distinctive path of the term M as in the proof of 3.4. We shall say that $\mathcal{M}$ has *uniform distinctive paths* if there exists a term P such that for any member M of $\mathcal{M}$ $P_M = P(\lambda\vec{x}.M)$.

In the rest of this paragraph we shall use this a bit changed denotation: we shall denote P_M, P_N etc. representations of paths while P will denote a term whose application to a term (i.e. PM) is the representation of a path. We shall use the "*If_then_else*" notation of terms, i.e. we shall write *If* B *then M else N* instead BMN, where B = T or B = F, from now to the end of this paper. This notation is more readable if terms are complex.

Proposition 5.1. *If the set $\mathcal{M} \subseteq \Lambda_f^T(\vec{x})$ has uniform distinctive paths lying in T then its closure has a complete discriminator.*

Proof. A complete discriminator for $\mathcal{M}$ is built analogously to the discriminator in 3.4. Only a small modification in $\tilde{\Delta}$ is needed, i.e. x is substituted by xy in the equation 3.1. Hence P_M becomes $P(\lambda\vec{x}.M)$ in equation 3.2. The rest of the construction remains unchanged.

Let Δ be a solution of the equation

$$\begin{array}{l}\Delta = \lambda uxyz.If\ Eq(B_1(xyu)y)(B_1(xyu)\,z)\\ \quad then\ (If\ Eq\,u\,(L(xy))\\ \quad\quad then\ \mathsf{T}\\ \quad\quad else\ (\Delta(R^+u)(xy)yz))\\ \quad else\ \mathsf{F}.\end{array}$$

Hence

$$\begin{array}{l}\Delta uP(\lambda\vec{x}.M)(\lambda\vec{x}.N) = If\ Eq(B_1(P(\lambda\vec{x}.M)u)\\ \quad(\lambda\vec{x}.M))(B_1(P(\lambda\vec{x}.M)u)\,(\lambda\vec{x}.N))\\ \quad then\ (If\ Eq\,u\,(L(P\lambda\vec{x}.M))\\ \quad\quad then\ \mathsf{T}\\ \quad\quad else\ (\Delta(R^+u)(P\lambda\vec{x}.M)(\lambda\vec{x}.M)(\lambda\vec{x}.N)))\\ \quad else\ \mathsf{F}.\end{array}$$

This is equation 3.2, since $P(\lambda\vec{x}.M) = P_M = \tilde{P}$. The rest of the proof is equal to the proof of 3.2 and 3.4.

□

Our next aim is to obtain some results concerning separability of terms. We start with some results which are instances of the definition of separability and don't concern distinctive paths.

$\mathcal{M} \subseteq \Lambda(\vec{x})$ is r.e. (is an enumeration) iff it is uniform, i.e. a sequence in the λ–calculus. A sequence is trivially r.e. For proving the other direction use the fact that there exists a term $E_{\vec{x}}$ such that $E_{\vec{x}}\ulcorner M\urcorner = M$ for any $M \in \Lambda(\vec{x})$. If $\chi : \mathbb{N} \to \{\natural M | M \in \mathcal{M}\}$ is a bijection onto and Ξ represents χ in λ-calculus, then the generator of the sequence is $\lambda y.E_{\vec{x}}(\Xi y)$.

Proposition 5.2. *A separable sequence is a numeral system.*

Proof. Let $\mathcal{M}$ be a sequence having M as generator, i.e. $\mathcal{M} = \{M_n | n \in \mathbb{N}\} = \{M\ulcorner n\urcorner | n \in \mathbb{N}\}$. Let S be a separator of the sequence. We shall construct an inverse of S on $\mathcal{M}$, i.e. a term S^{-1} such that for all $n \in \mathbb{N}$ $S^{-1}(SM_n) = M_n$ is available.

Let $SM_n = \ulcorner k\urcorner$. An algorithm for computing the inverse of S on $\mathcal{M}$ is as follows. Fix k and calculate $M\ulcorner i\urcorner, i = 0, 1, \ldots$ while $S(M\ulcorner i\urcorner) \neq \ulcorner k\urcorner$. The algorithm returns n such that $S(M\ulcorner n\urcorner) = \ulcorner k\urcorner$. S^{-1} is a term representing the described algorithm. More precisely $S^{-1}\ulcorner k\urcorner = M\ulcorner n\urcorner$. More formally: let F be a solution of the equation

$$F = \lambda xy.If\ Eq(Mx)y then\ulcorner 0\urcorner else R^+(F(R^+x)y).$$

Then $S^{-1} = \lambda x.M(F\ulcorner 0\urcorner x)$ (see the proof of 5.8 for more details). Now we can build terms $\mathbf{Zero}_{\mathcal{M}}, R^+_{\mathcal{M}}$ and $R^-_{\mathcal{M}}$ such that:

$$\begin{array}{lll}\mathbf{Zero}_{\mathcal{M}} & = \begin{cases}\mathsf{T} & \text{if}\quad SM_n = \ulcorner 0\urcorner,\\ \mathsf{F} & \text{otherwise},\end{cases} & \\ R^+_{\mathcal{M}}M_n & = M_m & \text{where}\quad R^+(SM_n) = SM_m,\\ R^-_{\mathcal{M}}M_n & = M_m & \text{where}\quad R^-(SM_n) = SM_m,\end{array}$$

where R^+ and R^- are as usual. In fact

$$\begin{array}{ll}\mathbf{Zero}_{\mathcal{M}} & = \lambda x.\mathbf{Zero}(Sx),\\ R^+_{\mathcal{M}} & = \lambda x.S^{-1}(R^+(Sx)),\\ R^-_{\mathcal{M}} & = \lambda x.S^{-1}(R^-(Sx)),\end{array}$$

where **Zero**, R^+ and R^- are as usual.

□

A term M is *left invertible* iff there exists a term M_L^{-1} such that $M_L^{-1} \circ M = \lambda x.M_L^{-1}(Mx) = I$. A weaker notion would be as follows: a term M has a *left inverse on numerals* if

$$\exists L \in \Lambda\,\forall n \in \mathbb{N}\,L(M\ulcorner n\urcorner) = \ulcorner n\urcorner.$$

It is known that a sequence whose generator has a left inverse is a numeral system (see [11]). In fact this property is enough since $\ulcorner n\urcorner = L(M\ulcorner n\urcorner) = LM_n$, i.e. the sequence M_n, $n = 0, 1, \ldots$ is separable, hence it is a numeral system by 5.2. Obviously L represents a bijective map from $\mathcal{M} = \{M_1, \ldots\}$ to numerals. It is well known that there exists a bijective map between any of two numeral systems (adequate numeral system in Barendregt's terminology), i.e. a term H such that $H\ulcorner n\urcorner = M_n$ and $H^{-1}M_n = \ulcorner n\urcorner$. (see [1] p.140 for the details). Hence the notion of left inverse on numerals coincides with the notion of bijective map on numerals.

We recapitulate the last results in a proposition as follows.

Proposition 5.3. *The next assertions are equivalent for a set of terms $\mathcal{M}$:*

1. *$\mathcal{M}$ is a numeral system*
2. *$\mathcal{M}$ is a separable sequence.*
3. *$\mathcal{M}$ is a sequence such that its generator is a bijection on numerals.*
4. *$\mathcal{M}$ is a sequence such that its generator has a left inverse on numerals.*

Example 3.6 shows that there exist separable sets that are not numeral systems. The set $\mathcal{M} = \{\lambda z.z(\lambda xy.f^n x)M'_n | n \in \mathbb{N}\}$ is separable but it is not a numeral system if $\mathcal{M}' = \{M'_n | n \in \mathbb{N}\}$ is not a sequence. In fact the existence of a successor for $\mathcal{M}$ would imply the existence of a generator for $\mathcal{M}'$ (the second projection of the iteration of the successor).

Definition 5.4. *A labeled path p^l is the labeled distinctive path for the term M in $\mathcal{M}$ if its underlying path p is the distinctive path for M in $\mathcal{M}$ and each of its nodes α is labeled by the $\sim_\alpha$ −equivalence class number eqn_α of M.*

Remark 5.5. The definition of the labeled distinctive path is relative to transformation since the definition of eqn_α is such.

We shall say $\mathcal{M} \subset \Lambda$ has *labeled uniform distinctive paths* if there exists a term P^l such that for any $M \in \mathcal{M}$ $P^l M$ is the labeled distinctive path for M.

Corollary 5.6. *If $\mathcal{M} \subset \Lambda_f^T(\vec{x})$, and it has uniform distinctive paths lying in T, then it has labeled uniform distinctive paths.*

Proof. Let us denote $\langle M, N\rangle$ an ordered pair of M and N in the λ−calculus (for example $\lambda x.xMN$). Since there exists a term P such that, for any $M \in \mathcal{M}$, PM is the distinctive path of M, P^l is as follows.

$$P^l = \lambda xy.\langle Pxy, B_1(Pxy)x\rangle,$$

where B_1 is defined in 2.17

□

Warning. The r.e. variant of preceding corollary fails since a set of distinctive paths can be r.e., while the corresponding set of labeled distinctive paths may not.

We shall denote p_M resp. p_N the distinctive paths of terms M resp. N. Their representatives in λ−calculus will be terms P_M resp. P_N. Analogously we shall denote p^l_M resp. p^l_N the labeled distinctive paths of terms M resp. N and P^l_M resp. P^l_N their representatives in λ−calculus.

Lemma 5.7. *1. There exists a test of equality of paths, i.e a term Eqp such that*

$$Eqp\, P_M P_N = \begin{cases} \mathsf{T} & p_M = p_N, \\ \mathsf{F} & p_M \neq p_N. \end{cases}$$

2. There exists a test of equality of labeled paths, i.e a term Eqp^l such that

$$Eqp^l P^l_M P^l_N = \begin{cases} \mathsf{T} & p^l_M = p^l_N, \\ \mathsf{F} & p^l_M \neq p^l_N. \end{cases}$$

Proof.

1. Remember $P_M\ulcorner i\urcorner = *$ iff $p_M i = *$. Since $\mathcal{N} \cup \{*\}$ is separable there exists a test of validity of the equation $P_M\ulcorner i\urcorner = *$. The algorithm for equality of paths is now straightforward: compare $P_M\ulcorner i\urcorner$ and $P_N\ulcorner i\urcorner$ for $i = 0, 1, \ldots, n$ where n is the minimal such that $P_M\ulcorner n\urcorner = *$ or $P_N\ulcorner n\urcorner = *$. Eqp is a term representing the preceding algorithm in the λ−calculus.
2. Analogously as in (1). At each step we have to test not only equality of nodes but equality of labels too.

□

Proposition 5.8. *If $\mathcal{M} \subseteq \Lambda_f^T(\vec{x})$ has a r.e. set of labeled uniform distinctive paths such that the corresponding underlying paths lie in T, then $\mathcal{M}$ is separable.*

Proof. Let $P_M^l = P^l M$ be the distinctive labeled path of the term $M \in \mathcal{M}$. Since $\mathcal{M}$ is r.e., there exists a general recursive function θ^l mapping $\mathbb{N}$ onto $\{\natural(P^l M) | M \in \mathcal{M}\}$ We can suppose that θ^l is one–to–one if $\mathcal{M}$ is infinite or it is one–to–one from $\{0, \ldots, m\}$ to the set of paths for an appropriate m if it is finite. In the following we shall omit the finite case since it can be treated analogously to the infinite one.

We can suppose that the numerals are closed normal forms. Since $P_M^l \ulcorner i \urcorner$ can be reduced to a closed normal form we can suppose that P_M^l is a closed term. Let Θ^l represent θ^l in the λ–calculus, hence for any $n \in \mathbb{N}$ $\Theta^l \ulcorner n \urcorner = \ulcorner P_M^l \urcorner$ for some $M \in \mathcal{M}$. Remember that there exists a term E such that for any closed term M the equation $E \ulcorner M \urcorner = M$ is accomplished. Hence $E(\Theta^l \ulcorner n \urcorner) = P_M^l$.

The idea of the proof is as follows: find a number ν so that the ν'th labeled path is equal to the labeled distinctive path of the term M, i.e. $Eqp^l(P^l M)(E(\Theta^l \ulcorner \nu \urcorner)) = \mathsf{T}$ (see Appendix). Let $\tilde{F}$ be a solution of the following equation.

$$\tilde{F} = \lambda xy. If\ Eqp^l(P^l y)(E(\Theta^l x)) \quad then\ \ulcorner 0 \urcorner\ else\ R^+(\tilde{F}(R^+ x)y).$$

Denote $F = \tilde{F}\ulcorner 0 \urcorner$. Hence $FM = \ulcorner \nu \urcorner$. It follows that the term $F = \tilde{F}\ulcorner 0 \urcorner$ is a separator for the set $\mathcal{M}$ as proved in the Appendix.

□

The theorem is not true in the other direction as shown by the following example.

Example 5.9. *Let $f : \mathbb{N} \to \mathbb{N}$ be a general recursive function which is onto and such that each $\mathbb{N}_k = \{i | f(i) = k\}$ is infinite. Choose exactly one number from each $\mathbb{N}_k$ such that the set of chosen numbers, say $\widetilde{\mathbb{N}}$, is not r.e. Suppose that there exists a bijection between $\mathbb{N}$ and a set of labeled distinctive paths. The set of Church numerals assigned to the the set $\widetilde{\mathbb{N}} \subset \mathbb{N}$ is separable by F – the term representing f in the λ-calculus. It is easy to see that the set of labeled distinctive paths is not r.e.*

An open problem is if the situation in the previous example is in some sense characteristic, i.e. if any separable set can be embedded into a separable set having a r.e. set of distinctive paths. Combining 5.6 and 5.8 we obtain the following proposition.

Proposition 5.10. *Let us suppose $\mathcal{M} \subseteq \Lambda_f^T(\vec{x})$ has a set of uniform distinctive paths lying in T, so that the corresponding set of labeled distinctive paths is r.e. Then $\mathcal{M}$ is separable.*

Proof. 5.6 implies $\mathcal{M}$ has labeled uniform distinctive paths. Since the set of labeled uniform distinctive paths is r.e. the assertion is implied by 5.8.

□

Since a separable sequence is a numeral system the following corollary is an immediate instance of the preceding proposition.

Corollary 5.11. *Let a sequence $\mathcal{M} = \{M_n | n \in \mathbb{N}\}$ have a set of uniform distinctive path lying in T so that the corresponding set of labeled distinctive path is r.e. Then $\mathcal{M}$ is a numeral system.*

Appendix

In the appendix we shall construct some terms that we used in the paper.

Let us denote Eq an equality test for the numeral sysytem, i.e.

$$Eq \ulcorner m \urcorner \ulcorner n \urcorner = \begin{cases} \mathsf{T}, & \text{if } m = n \\ \mathsf{F}, & \text{if } m \neq n. \end{cases} \tag{5.1}$$

If, for example Eq is defined as follows:

$$Eq \equiv \Theta(\lambda xyz.\mathbf{Zero}\, y(\mathbf{Zero}\, z\mathsf{TF}) \quad (\mathbf{Zero}\, z\, \mathsf{F}(x(R^- y)(R^- z)))),$$

then the equation

$$Eq \ulcorner m \urcorner \ulcorner n \urcorner = \mathbf{Zero} \ulcorner m \urcorner (\mathbf{Zero} \ulcorner n \urcorner \mathsf{TF}) \quad (\mathbf{Zero} \ulcorner n \urcorner\, \mathsf{F}(Eq(R^- \ulcorner m \urcorner)(R^- \ulcorner n \urcorner)))$$

is accomplished, hence Eq approaches 5.1.

On some places we use an algorithm counting steps until a condition is fulfilled, i.e.

$$n := 0;$$
$$\text{do} \quad n := n+1 \quad \text{whilenot} \quad \mathsf{Cond}(n).$$

A term representing the preceding algorithm is as follows. Let $\tilde{F}$ be a solution of the following equation.

$$\tilde{F} = \lambda x. \mathit{If}\ \mathit{Cond}\, x\ \mathit{then}\ \ulcorner 0 \urcorner\ \mathit{else}\, R^+(\tilde{F}(R^+x)),$$

where *Cond* is a Boolean term representing the condition Cond, i.e. $\mathit{Cond}\ulcorner n \urcorner = \mathsf{T}$ if $\mathsf{Cond}(n)$ is true and $\mathit{Cond}\ulcorner n \urcorner = \mathsf{F}$ if $\mathsf{Cond}(n)$ is false. R^+ denotes successor. Define $F = \tilde{F}\ulcorner 0 \urcorner$. Suppose $\mathsf{Cond}(n)$ is false for $n < v$ and $\mathsf{Cond}(v)$ is true, for a $v \in \mathbb{N}$. Hence

$$\begin{aligned} F &= \tilde{F}\ulcorner 0 \urcorner = \\ &= \mathit{If}\ \mathit{Cond}\ulcorner 0 \urcorner\ \mathit{then}\ \ulcorner 0 \urcorner\ \mathit{else}\, R^+(\tilde{F}(\ulcorner 1 \urcorner)) \\ &= \mathit{If}\ \mathit{Cond}\ulcorner 0 \urcorner\ \mathit{then}\ \ulcorner 0 \urcorner\ \mathit{else} \\ &\qquad\qquad \mathit{If}\ \mathit{Cond}\ulcorner 1 \urcorner\ \mathit{then}\ \ulcorner 1 \urcorner\ \mathit{else} \\ &\qquad\qquad R^+R^+(\tilde{F}(\ulcorner 2 \urcorner)), \end{aligned}$$

etc.

Hence $F = \ulcorner v \urcorner$.

The first application of the preceding result is the construction of a term representing an algorithm which calculates the length of paths. We supposed that the set $\mathcal{N} \cup \{*\}$ is separable, hence there exists a test of equality for it, say Eq_*, i.e. $Eq_*MN = T$ if $M = N$ and $Eq_*MN = F$ if $M \neq N$, for any $M, N \in \mathcal{N} \cup \{*\}$. Let the term $\tilde{L}$ be a solution of the following equation.

$$\begin{aligned} \tilde{L} = \lambda xy. \mathit{If}\ & Eq_*((yx)*) \\ & \mathit{then}\ \ulcorner 0 \urcorner\ \mathit{else}\, R^+(\tilde{L}(R^+x)y). \end{aligned}$$

Hence

$$\begin{aligned} \tilde{L}\ulcorner 0 \urcorner P = \mathit{If}\ & Eq_*((P\ulcorner 0 \urcorner)*) \\ & \mathit{then}\ \ulcorner 0 \urcorner\ \mathit{else}\, R^+(\tilde{L}(\ulcorner 1 \urcorner)P). \end{aligned}$$

Obviously $\tilde{L}\ulcorner 0 \urcorner P$ counts the number of nodes lying on the (nonvoid) path represented by P. Hence $L = R^+\tilde{L}\ulcorner 0 \urcorner$ is a term that calculates the length of the path.

Using the notation from the proof of 5.8 a separator is constructed as follows. Let $\tilde{F}$ be a solution of the followin equation.

$$\begin{aligned} \tilde{F} = \lambda xy. \mathit{If}\ & Eqp^l(P^ly)(E(\Theta^lx)) \\ & \mathit{then}\ \ulcorner 0 \urcorner\ \mathit{else}\, R^+(\tilde{F}(R^+x)y). \end{aligned}$$

Denote $F = \tilde{F}\ulcorner 0 \urcorner$. Hence

$$FM = \tilde{F}\ulcorner 0 \urcorner M = \ulcorner v \urcorner,$$

presuming the v'th labeled path is equal to the distinctive path P^lM. Hence the term $F = \tilde{F}\ulcorner 0 \urcorner$ is a separator for the set $\mathcal{M}$.

Acknowledgment

It is a pleasure to thank the referees for many valuable suggestions which resulted in an improvement of the manuscript.

References

[1] H.P. Barendregt, *The lambda calculus, its syntax and semantics*, Studies in Logic and The Foundations of Mathematics, vol. 103, North-Holland, 1984.

[2] Alessandro Berarducci and Benedetto Intrigila, *Some new results on easy lambda-terms*, Theoretical Computer Science **121** (1993), 71–88.

[3] C. Böhm, *Alcune proprietá delle forme $\beta-\eta-$normali nel $\lambda - k-$calcolo*, Pubblicationi dell' Instituto per Applicazioni del Calcolo n.696, Roma (19 pp.).

[4] ______, *An abstract approach to (hereditary) finite sequences of combinators*, To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, J.P. Seldin and J.R. Hindley, 1980, pp. 231–242.

[5] C. Böhm and M. Dezani-Ciancaglini, *Combinatory logic as monoids*, Fundamenta Informaticae **XII** (1989), 525–539.

[6] C. Böhm, A. Piperno, and E. Tronci, *Solving equations in lambda-calculus*, Logic Colloquium'88 (Ferro, Bonotto, Valentini, and Zanardo, eds.), Elsevier Science Publishers B.V.(North-Holland), 1989, pp. 139–160.

[7] Corrado Böhm and Enrico Tronci, *About systems of equations, x-separability, and left-invertibility in the $\lambda-$calculus*, Information and Computation **90** (1991), 1–32.

[8] M. Coppo, M. Dezani-Ciancaglini, and S. Ronchi Della Rocca, *(semi)-separability of finite sets of terms in Scott's D_∞-models of the $\lambda-$calculus*, Automata, Languages and Programming (Fifth Colloquium, Udine, Italy 1978), Lecture Notes in Computer Science, vol. 62, Springer Verlag, 1978, pp. 142–164.

[9] E.V. Krishnamurthy and B.P. Vickers, *Compact numeral representation with combinators*, The Journal of Symbolic Logic **52** (1987), no. 2, 519–525.

[10] Giuseppe Longo, *On the problem of deciding equality in partial combinatory algebras and in a formal system*, Studia Logica **4** (1976), 363–375.

[11] S. Ronchi Della Rocca, *Discriminability of infinite sets of terms in the D_∞−models of the λ−calculus*, Lecture Notes in Computer Science, vol. 112, 1981.

[12] Christopher P. Wadsworth, To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, ch. Some Unusual λ−Calculus Numeral Systems, pp. 215–230, J.P. Seldin and J.R. Hindley, 1980, pp. 215–230.

Received: March, 1993
Accepted: March, 1995

Contact address:
Faculty of Civil Engineering
University of Maribor
Smetanova 17, 2000 Maribor
Slovenia
e-mail: gaborovic.@uni-mb.si

SVITAN GABOROVIČ a lecturer of Mathematics received the B.Sc. from the University of Ljubljana and M.Sc. from the University of Zagreb. He joined the Faculty of Civil Engineering, University of Maribor after a 9 year career in industry. His research interests include category theory, lambda calculus, functional programming and semantics of programming languages.