Journal of Computing and Information Technology - CIT 20, 2012, 3, 201–207
doi:10.2498/cit.1002098

Generalized Models from $Beta(p,2)$ Densities with Strong Allee Effect: Dynamical Approach

Sandra M. Aleixo and J. Leonel Rocha
Mathematics Unit, ADM and CEAUL, Instituto Superior de Engenharia de Lisboa, Portugal

A dynamical approach to study the behaviour of generalized populational growth models from $Beta(p, 2)$ densities, with strong Allee effect, is presented. The dynamical analysis of the respective unimodal maps is performed using symbolic dynamics techniques. The complexity of the correspondent discrete dynamical systems is measured in terms of topological entropy. Different populational dynamics regimes are obtained when the intrinsic growth rates are modified: extinction, bistability, chaotic semistability and essential extinction.

Keywords: $Beta(p, 2)$ densities, Allee effect, symbolic dynamics, topological entropy

1. Introduction and Motivation

In previous studies, the authors presented a successful approach to study population growth models, proportional to $Beta(p, 2)$ densities, [1] and [2]. These models are natural extensions of the logistic Verhuslt growth model, originally introduced as a demographic model and often considered as the basis of modern chaos theory. The models presented have no Allee effect for some values of the parameter p, namely for $1 < p \leq 2$, but this drawback was corrected since the authors suggested suitable corrections, [2]. In [6], we generalized these models by adding a positive parameter C, which allows models to be more flexible with variable extinction rates. The case of $C = 0$ was studied in [2].

The generalized population growth rates or generalized models from $Beta(p, 2)$ densities with Allee effect, are defined by

$$f_i^*\left(N(t)\right) = r_i^* N(t)^{p-1}\left(1 - \frac{N(t)}{K}\right) T_i\left(N(t)\right) \tag{1}$$

with $r_i^* > 0$, $1 < p \leq 2$ and $i = 1, 2, 3$, where K is the carrying capacity and the correction factors that adjust the Allee effect are defined by

$$T_1\left(N(t)\right) = \frac{N(t) - E}{N(t) + C}$$

$$T_2\left(N(t)\right) = \frac{N(t) - E}{K + C}$$

$$T_3\left(N(t)\right) = \frac{N(t) - E}{E + C}$$

with $0 < E < K$ and $C > 0$, where E is the rare faction critical density or Allee limit. Consequently, the respective *per capita* growth rates are

$$g_i^*\left(N(t)\right) = \frac{f_i^*\left(N(t)\right)}{N(t)}.$$

We consider that an Allee effect occurs when the *per capita* growth rates increase at low densities, i.e., $\left(g_i^*\right)'\left(N(t)\right) > 0$ for $N(t)$ sufficiently small. A strong Allee effect is verified when there is a positive equilibrium density or Allee limit $E = E^u$, i.e.,

$$\lim_{N(t)\to E^u} f_i^*\left(N(t)\right) = 0$$

where E^u is an unstable equilibrium value, such that

$$g_i^*\left(N(t)\right) < 0 \Leftrightarrow 0 < N(t) < E^u \vee N(t) > K$$
$$g_i^*\left(N(t)\right) > 0 \Leftrightarrow E^u < N(t) < K \tag{2}$$

considering $0 < E^u < K, C > 0$ and $i = 1,2,3$. This means that the *per capita* growth rates increase with population size at low densities, indicating more deaths than births in the population, i.e., extinction occurs. We remark that, attending to (2), with $i = 1,2,3$, there is a subinterval $I_i \subset]0,K]$ such that the generalized *per capita* growth rates verify the inequality

$$g_i^* \left(\tilde{N}(t)\right) > (f_i^*)' (0) = -\infty, \ \forall \tilde{N}(t) \in I_i.$$

So, the models under study exhibit strong Allee effect, see [6], [7] and Figure 1.

In the present paper, we study families of unimodal maps, which represent the generalized models from $Beta(p,2)$ densities, given by Eq. (1), with three adjustment factors, which incorporate strong Allee effect. The families of maps considered are defined by

$$f_{i;r_i,p}(x) = r_i \, x^{p-1} (1-x) \ F_i(x) \qquad (3)$$

with $x = \frac{N(t)}{K} \in [0,1]$ the normalized population dimension size, $r_i = r_i^* K^{p-1} > 0$ an intrinsic growth rate of the number of individuals, $1 < p \leq 2$ a shape parameter and $i = 1,2,3$, where the adjustment factors are defined by

$$\begin{aligned} F_1(x) &= \frac{Kx - E}{Kx + C} \\ F_2(x) &= \frac{Kx - E}{K + C} \\ F_3(x) &= \frac{Kx - E}{E + C}. \end{aligned} \qquad (4)$$

We consider $A_{r_i} = f_{i;r_i,p}(A_{r_i})$ the first positive fixed point of each map $f_{i;r_i,p}$ and $A_{r_i}^* = max\{f_{i;r_i,p}^{-1}(A_{r_i})\}$, with $i = 1,2,3$. Under these assumptions, the maps (3) define a family of unimodal maps,

$$f_{i;r_i,p} : \left[A_{r_i}, A_{r_i}^*\right] \longrightarrow \left[A_{r_i}, A_{r_i}^*\right]$$

since they satisfy the following conditions:

- $f_{i;r_i,p}(A_{r_i}) = f_{i;r_i,p}\left(A_{r_i}^*\right) = A_{r_i}$;
- $f_{i;r_i,p} \in C^3\left(\left]A_{r_i}, A_{r_i}^*\right[\right)$;
- $f'_{i;r_i,p}(x) \neq 0, \forall x \in \left]A_{r_i}, A_{r_i}^*\right[\setminus \{c_i\}$, where c_i is the positive critical point of each $f_{i;r_i,p}(x)$;
- $f'_{i;r_i,p}(c_i) = 0$ and $f''_{i;r_i,p}(c_i) < 0$;
- the Schwarz derivative of $f_{i;r_i,p}(x)$ is given by

$$S\left(f_{i;r_i,p}(x)\right) = \frac{f'''_{i;r_i,p}(x)}{f'_{i;r_i,p}(x)} - \frac{3}{2}\left(\frac{f''_{i;r_i,p}(x)}{f'_{i;r_i,p}(x)}\right)^2.$$

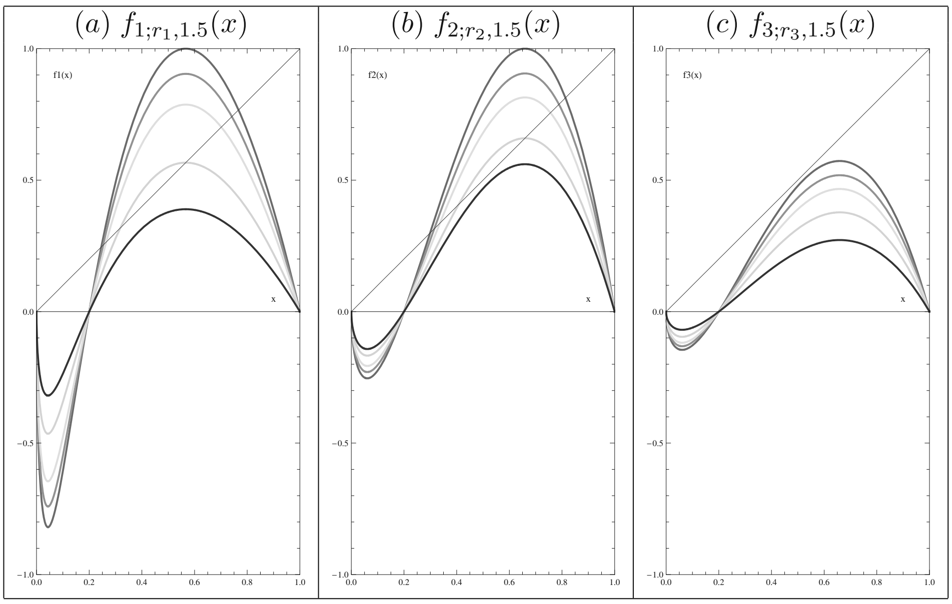

Figure 1. Population growth rates curves: with $E = 2$, $K = 10$, (a) and (b) with $C = 2.01$, (c) with $C = 5$, and several intrinsic growth rates.

It can be seen that $S\left(f_{i;r_i,p}(x)\right) < 0$, for all $x \in \left]A_{r_i}, A^*_{r_i}\right[\backslash\{c_i\}$, for each $i = 1, 2, 3$, and $S\left(f_{i;r_i,p}(c_i)\right) = -\infty$. This condition ensures a "good" dynamic behaviour of the models: continuity and monotonicity of topological entropy, order in the succession of bifurcations, the existence of an upper limit to the number of stable orbits and the non-existence of wandering intervals.

Therefore, in the following sections, the three families of unimodal maps defined by Eq. (3), satisfying the conditions presented above, will be studied from a dynamical systems viewpoint, using techniques of symbolic dynamics, see Figures 1 and 2.

2. Discrete Dynamical Systems: Symbolic Dynamics and Topological Entropy

Symbolic dynamics is a theory composed of a set of results, methods and techniques which have a primordial role in the study about qualitative and quantitative properties of discrete dynamical systems. The topological complexity of a dynamical system is usually measured by its topological entropy, see [3], [4], [5] and references therein.

We consider, for each value of the parameter r_i, the orbit of the critical point c_i for each map $f_{i;r_i,p}(x)$, with $i = 1, 2, 3$,

$$O_{r_i}(c_i) = \left\{x_k : x_k = f^k_{i;r_i,p}(c_i),\ k \in \mathbf{N}_0\right\}$$

defined by an iterative process, where

$$x_k = f^k_{i;r_i,p}(c_i) = f_{i;r_i,p}(x_{k-1}).$$

Thus, for each value of the intrinsic growth rate r_i, is considered the orbit of the density when each growth rate is maximum.

In order to study the topological properties of these orbits, we associate to each orbit $O_{r_i}(c_i)$ a sequence of symbols, corresponding to the critical point itinerary, denoted by

$$S^{(r_i)} = S^{(r_i)}_0 S^{(r_i)}_1 S^{(r_i)}_2 \ldots S^{(r_i)}_k \ldots,$$

with $k \in \mathbf{N}_0$, where $S^{(r_i)}_k$ belongs to the alphabet $\mathcal{A} = \{L, C, R\}$, with each symbol defined by

$$S^{(r_i)}_k = \begin{cases} L & \text{if } f^k_{i;r_i,p}(c_i) < c_i \\ C & \text{if } f^k_{i;r_i,p}(c_i) = c_i \\ R & \text{if } f^k_{i;r_i,p}(c_i) > c_i \end{cases}.$$

Note that the alphabet $\mathcal{A}$ is an ordered set of symbols corresponding to the intervals of monotonicity and to the critical points of the maps $f_{i;r_i,p}$. The space of all symbolic sequences of $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbf{N}}$.

The expansive maps admit Markov partitions, whose existence is implicit in the works of Bowen and Ruelle. In this study, we consider the existence of Markov partitions, which are characterized by the orbits of the critical points of the maps $f_{i;r_i,p}$. Consider the set of points corresponding to the k-periodic orbit or kneading sequence of the critical points

$$S^{(r_i)} = (CS^{(r_i)}_1 S^{(r_i)}_2 \ldots S^{(r_i)}_{k-1})^\infty \in \mathcal{A}^{\mathbf{N}}.$$

This set of points determines three Markov partitions of the intervals

$$I_i = \left[f^2_{i;r_i,p}(c_i), f_{i;r_i,p}(c_i)\right] \subseteq [A_i, A^*_i]$$

in a finite number of subintervals, denoted by $\mathcal{P}_{I_i} = \left\{I_{i,1}, I_{i,2}, \ldots, I_{i,k-1}\right\}$. The dynamics of each map $f_{i;r_i,p}$ is completely characterized by the symbolic sequence $S^{(r_i)}$ associated to the critical point itinerary.

For each map $f_{i;r_i,p}$ and Markov partition associated is induced a subshift of finite type whose Markov transition matrix $B_i = [b_{mn}]$, $(k-1) \times (k-1)$, is defined by

$$b_{mn} = \begin{cases} 1, & \text{if } int\left(I_{i,n}\right) \subseteq f_{i;r_i,p}\left(int\left(I_{i,m}\right)\right) \\ 0, & \text{otherwise} \end{cases}.$$

Usually, the subshift is denoted by $\left(\sum_{B_i}, \sigma\right)$, where σ is a shift map in $\sum^{\mathbf{N}}_{k-1}$ defined by $\sigma\left(S_1 S_2 \ldots\right) = S_2 S_3 \ldots$, where $k-1$ subshifts states are $\sum_{k-1} = \{1, \ldots, k-1\}$.

The topological entropy of each map $f_{i;r_i,p}$, in the phases space, is defined in the associated symbolic space as the asymptotic growth rate

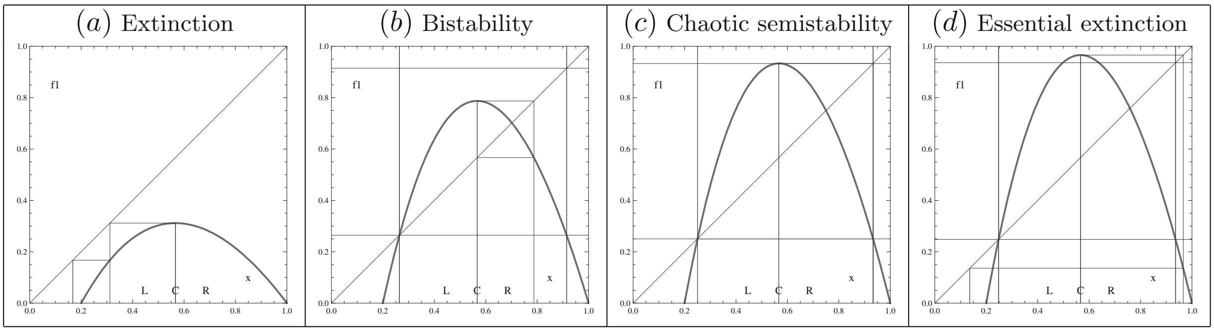

Figure 2. Population growth rates curves $f_{1;r_1,1.5}(x)$: with $E = 2$, $K = 10$, $C = 2.01$, (a) CL^∞ with $r_1 = 2$, (b)$(CR)^\infty$ with $r_1 = 5.053$, (c) $CRA_{r_1}^\infty$ with $r_1 = 5.99$, (d) CRL^∞ with $r_1 = 6.2$.

of the admissible words (finite symbolic sequences) in relation to the length of the words, i.e.,

$$h_{top}\left(f_{i;r_i,p}\right) = \lim_{n\to\infty} \frac{\ln N(n)}{n}$$

where $N(n)$ is the number of admissible words of length n. For a subshift of finite type, unidirectional or bidirectional, described by the Markov transition matrix B_i, the topological entropy is given by $h_{top}(\sigma) = \ln(\lambda_{B_i})$, where λ_{B_i} is the spectral radius of the transition matrix B_i. For a more detailed approach about subshifts of finite type and the Perron-Frobenius Theorem for Markov transition matrix, see [3], [4] and references therein.

3. Extinction, Bistability, Chaotic Semistability and Essential Extinction

This section is devoted to the study of the complex dynamical behaviour of the generalized models from $Beta(p,2)$ densities, with strong Allee effect. The complexity of these models, described by the maps $f_{i;r_i,p}$, defined in (3), is displayed as a function of the parameters r_i and p. Or, equivalently, the dynamical behaviour is analyzed according to the intrinsic growth rate of the number of individuals and the shape parameter p, related with the growth-retardation phenomena. This complexity is measured in terms of topological entropy. The parameter space is split into different regions, according to the dynamical behaviour of the models. The dynamics of the models $f_{i;r_i,p}$ with strong Allee effect fall generically into four categories: extinction, bistability, chaotic semistability and essential extinction, Figure 4.

If the *per capita* growth rate is less than one for all densities, or in an equivalent way, if $f_i^*(N(t)) < N(t)$, $\forall\, N(t) > 0$, then extinction occurs for all initial population densities. In this case, the iterates of the maps $f_{i;r_i,p}$ are always attracted to the "extinction area", Figures 1(c) and 2(a). For each $i = 1,2,3$ and $1 < p \leq 2$, there are values of the intrinsic growth rates defined by

$$0 < r_i < r_i(p) = a_{r_i}^{2-p}(1-a_{r_i})^{-1}F_i^{-1}(a_{r_i})$$

where F_i is like in (4) and a_{r_i} is the only positive fixed point of each map, such that

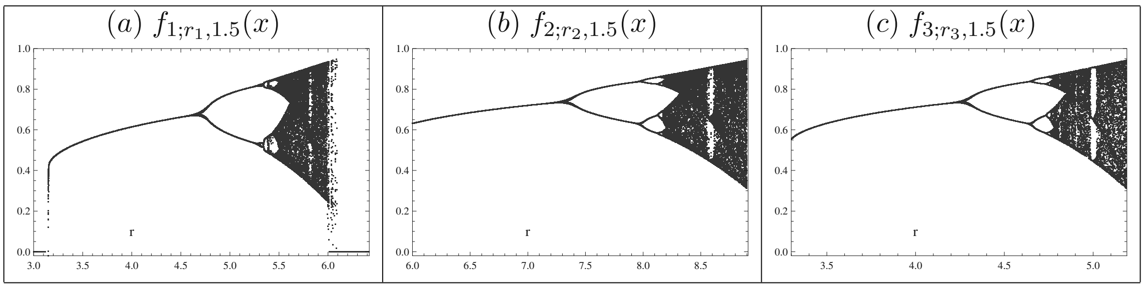

Figure 3. Bifurcation diagrams: with $E = 2$, $K = 10$, (a) and (b) with $C = 2.01$, (c) with $C = 5$.

$f_{i;r_i,p}(c_i) < c_i$, where c_i is the critical point of each family of maps. In this region, the symbolic sequences associated to the critical points orbits of these families of maps are of the type CL^∞, Figure 2(a), and its topological entropy is null, [3] and [5]. In this case, the negative Schwartzian derivative hypothesis is necessary to ensure that almost every initial density goes to extinction. Without this condition, we are only ensured that initial densities near c_i conduct to extinction and other initial densities may lead to stable periodic points.

Otherwise, if the *per capita* growth rate is greater than one for intermediate values densities, then we are dealing with extinction survival or unconditional survival scenarios, see [6].

Definition 1. *For the values of the parameters $1 < p \le 2$ and $r_i > 0$, with $i = 1,2,3$, the smallest positive fixed point A_{r_i}, that verifies $f^2_{i;r_i,p}(c_i) < A_{r_i}$, where c_i is the critical point of each family of unimodal maps, is designated by Allee point.*

A behaviour of bistability is defined when a population persists for intermediate initial densities and otherwise goes extinct, [7] and Figure 2(b). This is, if $f^2_{i;r_i,p}(c_i) > A_{r_i}$ then

$$f^n_{i;r_i,p}(x) \ge A_{r_i},\ \forall\, x \in \left[A_{r_i}, A^*_{r_i}\right], n \ge 0$$

and

$$\lim_{n\to\infty} f^n_{i;r_i,p}(x) = 0,\ \forall x \notin \left[A_{r_i}, A^*_{r_i}\right].$$

For the parameter values, such that these conditions hold, the associated dynamic behaviour is characterized by bifurcations (period doubling) and chaos, verifying Sharkovsky's Theorem, Figure 3. The period doubling region (where the topological entropy is null, [1], [3] and [5]) starts with the symbolic sequence $(CR)^\infty$, Figure 2(b), and finishes with the symbolic sequence $\left(CRLR^3\right)^\infty$, whose transition matrix is

$$B_i = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the spectral radius is $\lambda_{B_i} = 1.272\ldots$. The topological entropy becomes positive and it is a non-decreasing function in order to the parameter r_i, [1], [4] and [5]. In turn, the chaotic region ends at the designated "fullshift", whose symbolic sequence is CRL^∞. These results are illustrated in Table 1.

The parameters for which the bistability region finishes, or, equivalently, where the chaotic region ends, characterize a line designated by chaotic semistability, Figure 2(c). This line is defined by the parameters such that $f^2_{i;r_i,p}(c_i) = A_{r_i}$. In the chaotic semistability line, the dynamical behaviour of each $f_{i;r_i,p}$, restricted to $\left[A_{r_i}, A^*_{r_i}\right]$, is chaotic. Thus, we are in the "fullshift" line, whose symbolic sequence is $CRA^\infty_{r_i}$ and the topological entropy is $\ln 2$. For all densities outside the invariant interval $\left[A_{r_i}, A^*_{r_i}\right]$, the populations become extinct, Figure 2(c).

After this line, we have an essential extinction region in which "almost every" initial density leads to extinction. The maps with parameters in this region are characterized by $f^2_{i;r_i,p}(c_i) < A_{r_i}$ and the iterates for Lebesgue almost every $x \in [0,1]$ goes to "extinction area" $\left[0, \frac{E}{K}\right[$, Figure 2(d). The symbolic sequences are of the type CRL^∞ and the topological entropy is $\ln 2$. At this stage, Cantor sets become observable, [1].

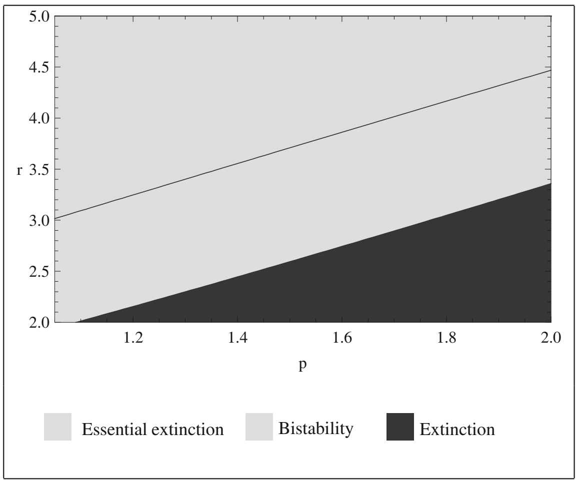

Figure 4. Extinction, bistability, chaotic semistability and essential extinction regions for $f_{3;r3,p}(x)$, with $E = 2$, $K = 10$ and $C = 3$.

Table 1 shows the numerical results that illustrate the application of iteration theory and symbolic dynamics, to generalized models from $Beta(p,2)$ densities, with strong Allee effect. It

k	$S^{(ri)}$	$f_{1;r_1,1.5}(x)$ $C=0$	$f_{1;r_1,1.5}(x)$ $C=2.01$	$f_{1;r_1,1.5}(x)$ $C=3$	$f_{2;r_2,1.5}(x)$ $C=0$	$f_{2;r_2,1.5}(x)$ $C=2.01$	$f_{2;r_2,1.5}(x)$ $C=3$	$f_{3;r_3,1.5}(x)$ $C=0$	$f_{3;r_3,1.5}(x)$ $C=2.01$	$f_{3;r_3,1.5}(x)$ $C=3$	h_{top}
2	$(CR)^\infty$	3.670	5.053	5.717	6.411	7.699	8.334	1.282	2.570	3.205	0.000
4	$(CRLR)^\infty$	3.934	5.384	6.080	6.716	8.066	8.731	1.343	2.693	3.358	0.000
8	$(CRLR^3LR)^\infty$	3.990	5.456	6.158	6.783	8.146	8.818	1.356	2.720	3.391	0.000
6	$(CRLR^3)^\infty$	4.063	5.548	6.260	6.870	8.251	8.932	1.374	2.755	3.435	0.241
8	$(CRLR^5)^\infty$	4.098	5.593	6.309	6.911	8.301	8.985	1.382	2.771	3.455	0.304
7	$(CRLR^4)^\infty$	4.142	5.646	6.367	6.957	8.356	9.045	1.391	2.790	3.478	0.382
5	$(CRLR^2)^\infty$	4.177	5.693	6.419	7.004	8.420	9.105	1.400	2.808	3.502	0.414
7	$(CRLR^2LR)^\infty$	4.207	5.736	6.467	7.050	8.467	9.165	1.410	2.827	3.525	0.442
8	$(CRLR^2LR^2)^\infty$	4.231	5.768	6.504	7.085	8.509	9.210	1.417	2.841	3.542	0.468
3	$(CRL)^\infty$	4.254	5.804	6.545	7.129	8.562	9.268	1.425	2.858	3.564	0.481
6	$(CRL^2RL)^\infty$	4.261	5.818	6.561	7.149	8.587	9.294	1.429	2.867	3.574	0.481
8	$(CRL^2RLR^2)^\infty$	4.280	5.847	6.595	7.188	8.633	9.344	1.437	2.882	3.594	0.499
7	$(CRL^2RLR)^\infty$	4.293	5.866	6.616	7.209	8.658	9.372	1.441	2.891	3.604	0.522
8	$(CRL^2RLRL)^\infty$	4.303	5.882	6.634	7.229	8.682	9.397	1.445	2.898	3.614	0.539
5	$(CRL^2R)^\infty$	4.307	5.888	6.642	7.239	8.694	9.411	1.447	2.902	3.619	0.544
8	$(CRL^2R^3L)^\infty$	4.311	5.895	6.651	7.249	8.706	9.424	1.449	2.907	3.624	0.547
7	$(CRL^2R^3)^\infty$	4.319	5.907	6.664	7.264	8.724	9.443	1.452	2.912	3.632	0.562
8	$(CRL^2R^4)^\infty$	4.326	5.918	6.676	7.275	8.737	9.458	1.455	2.917	3.637	0.574
6	$(CRL^2R^2)^\infty$	4.331	5.926	6.686	7.286	8.750	9.472	1.457	2.921	3.643	0.584
8	$(CRL^2R^2LR)^\infty$	4.335	5.933	6.694	7.296	8.763	9.487	1.459	2.926	3.648	0.591
7	$(CRL^2R^2L)^\infty$	4.339	5.941	6.703	7.307	8.776	9.500	1.461	2.930	3.653	0.601
4	$(CRL^2)^\infty$	4.343	5.950	6.715	7.325	8.798	9.523	1.465	2.937	3.662	0.609
8	$(CRL^3RL^2)^\infty$	4.344	5.951	6.716	7.327	8.800	9.525	1.466	2.938	3.663	0.609
7	$(CRL^3RL)^\infty$	4.347	5.958	6.725	7.343	8.819	9.546	1.468	2.944	3.671	0.618
8	$(CRL^3RLR)^\infty$	4.350	5.963	6.731	7.352	8.830	9.558	1.470	2.948	3.676	0.626
6	$(CRL^3R)^\infty$	4.352	5.968	6.736	7.360	8.839	9.568	1.472	2.951	3.680	0.633
8	$(CRL^3R^3)^\infty$	4.354	5.971	6.741	7.367	8.848	9.577	1.473	2.954	3.683	0.639
	CRA_{ri}^∞	4.360	5.988	6.763	7.379	8.908	9.639	1.483	2.974	3.707	ln 2
	CRL^∞	4.694	6.418	7.240	7.870	9.452	10.231	1.574	3.156	3.935	ln 2

Table 1. Topological order in bistability, chaotic semistability and essential extinction regions: symbolic sequences, intrinsic growth rates and topological entropy, with $K=10$ and $E=2$.

establishes a topological order to each model, depending on the intrinsic growth rate r_i, which reflects the bistability, chaotic semistability and essential extinction behaviours of the population growths. Analyzing the results, it was concluded that:

- Flexibility in the extinction rate: increasing the parameter C in each model, the intrinsic growth rate r_i increases in the extinction region, and consequently also increase the intrinsic growth rates in bistability, chaotic semistability and essential extinction regions.
- Monotonicity of the topological entropy: if the intrinsic growth rate r_i increases, then the topological entropy is a non-decreasing function in order to r_i, (consequence of the negative Schwartzian derivative), [4] and [5].
- Isentropic curves: for each of the models, there is an intrinsic growth rate r_i such that the topological entropy is equal.

4. Aknowledgments

This research has been supported by National Funds through FCT – Fundação para a Ciência e a Tecnologia, project PEst-OE/MAT/ UI0006/ 2011, and PTDC/FEDER.

References

[1] S. M. ALEIXO, J. L. ROCHA, D. D. PESTANA, Dynamical behaviour in the parameter space: new populational growth models proportional to beta densities. In *Proc. of ITI 2009, 31th Int. Conf. Information Technology Interfaces*, 213–218.

[2] S. M. ALEIXO, J. L. ROCHA, D. D. PESTANA, Populational growth models proportional to beta densities with Allee effect. *Am. Inst. Phys.*, 1124 (2009), 3–12.

[3] D. LIND, B. MARCUS, *An Introduction to Symbolic Dynamics and Codings*. Cambridge Univ. Press, Cambridge 1995.

[4] W. MELO, S. STRIEN, *One Dimensional Dynamics*. Springer, New York, 1993.

[5] J. MILNOR, W. THURSTON, On iterated maps of the interval. *Lect. Notes in Math*, 1342 (1988), 465–563. Springer-Verlag.

[6] J. L. ROCHA, S. M. ALEIXO, Modeling Allee effect from $Beta(p,2)$ densities. In *Proc. of ITI 2012, 34th Int. Conf. Information Technology Interfaces*, pp. 461–466.

[7] S. J. SCHREIBER, Allee effects, extinctions, and chaotic transients in simple population models. *Theor. Popul. Biol.*, 64 (2003), 201–209.

Received: June, 2012
Accepted: August, 2012

Contact addresses:

Sandra M. Aleixo
Mathematics Unit
ADM and CEAUL
Instituto Superior de Engenharia de Lisboa
Rua Conselheiro Emídio Navarro, 1
1959-007 Lisboa
Portugal
e-mail: sandra.aleixo@adm.isel.pt

J. Leonel Rocha
Mathematics Unit
ADM and CEAUL
Instituto Superior de Engenharia de Lisboa
Rua Conselheiro Emídio Navarro, 1
1959-007 Lisboa
Portugal
e-mail: jrocha@adm.isel.pt

SANDRA M. ALEIXO is Coordinator Professor at the Department of Mathematics of the Superior Institute of Engineering of Lisbon. She graduated from the Faculty of Sciences of the University of Lisbon, in the field of probability, statistics and operational research: B.Sc. in 1992, M.Sc. in 1995 and Ph.D. in 2008. As member of the Research Center of Statistics and Applications at the University of Lisbon, she has been involved in research on her interest areas: data modeling, probability models, sampling randomly stopped sums, simulation and dynamical systems and applications. Lately, she has published papers related to population growth models, some of them associated with certain probability distributions, which were analyzed in terms of discrete dynamical systems. The existence of Allee effect was studied for some models.

J. LEONEL ROCHA received his Diploma in 1994, the Masters degree in 1998 and the Ph.D. degree in 2004, all in mathematics, the last ones were obtained from the Technical University of Lisbon, School of Engineering (IST). Since 1997, his research has been centered on dynamical systems: topological and metrical invariants, symbolic dynamics, bifurcations and chaos. He continued research in applied mathematics in the Center of Statistics and Applications of the University of Lisbon. His interests are in the areas of differential equations (growth models), networks, synchronization and applications.